Fixed point theorems for R''-Kanan mapping in b-metric spaces

Cholatis Suanoom[1,*], **Chatsuda Chanmanee**[2] **and Patchara Muangkarn**[3]

Program of Mathematics, Faculty of Science and Technology, Kamphaengphet Rajabhat University, Kamphaengphet, Thailand; [1]cholatis.suanoom@gmail.com, [2]chatsuda.chanmanee@gmail.com; [3]putchara_31@windowslive.com
[*]Correspondence: cholatis.suanoom@gmail.com

Abstract: In this paper, we extend and improve R'-contractions and via R'-functions mappings to R'-Max-kanan and R''-kanan mappings by using the concept of kanan mappings. Second, we establish new mapping, that is R'-Max-kanan and R''-kanan mappings and prove the results of fixed point for R'-Max-kanan and R''-kanan mappings in b-metric spaces. Moreover, we obtain fixed point theorems for R'-Max-kanan and R''-kanan mappings in b-metric spaces and present some examples to illustrate and support our results.

Keywords: fixed point; b-metric spaces; R'-Max-kanan mappings; R''-kanan mappings; R-function; K-simu lation function
AMS Math Classification (2010) : 47H05, 47H10, 47J25

---

1 Introduction and Preliminaries

The Banach fixed point theorem is an important tool in the theory of metric spaces, it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points was introduced by Stefan Banach [1] in 1922. Important, the study of fixed point theory started from such theorem as follows: Let T be a self-mapping on metric spaces X. Then T is called a *contraction mapping* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y), \quad \text{for all } x, y \in X.$$

In 1969, Kannan [2] extended the concept of Banach [1] and obtained the same conclusion as in Banach's Theorem but with different sufficient conditions as follows: Let T be a self-mapping on metric spaces X. Then T is called a *Kannan mapping* if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty), \quad \text{for all } x, y \in X.$$

In 1972, Bianchini [3] introduced generalized Kannan mapping which generalized the concept of Kannan [2] as follows: Let T be a self-mapping on metric spaces X. Then T is called a *generalized Kannan mapping or Bianchini mapping* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\}, \quad \text{for all } x, y \in X.$$

In 2015, Khojasteh et al. [4] introduced the notion of Z-contraction defined by simulation function. Then, Khojasteh et al. proved a new fixed point theorem concerning Z-contraction which generalizes Banach's contraction principle. Recently, Roldán López de Hierro and Shahzad [5] introduced the concept of R-contraction defined by R-function in order to generalize the previous results.

On the other hand, Bakhtin [6] and Czerwik [7] developed the notion of b-metric space and established some fixed point theorems in b-metric spaces. Subsequently, several results appeared in this direction [8, 9, 10, 11, 12, 13, 14, 15]. Recently, Mongkolkeha and et al. [17] introduced the notion of a simulation function in the setting of b-metric spaces as follows:

Definition 1.1. *[7] A b-metric on a set X is a mapping $d : X \times X \to [0, +\infty)$ satisfying the following conditions: for any $x, y, z \in X$,*
(b_1) $d(x, y) = 0$ *if and only if* $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) *there exists* $K \geq 1$ *such that* $d(x, y) \leq K(d(x, z) + d(z, y))$.
Then (X, d) is known as a b-metric space with coefficient K.

Note that every metric space is a b-metric space with $K = 1$. Some examples of b-metric space are given below:

Example 1.1. [7, 16]

(i) Let $X = \mathbb{R}$. Define a mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = (x + y)^2 \ for \ all \ \ x, y \in X.$$

Then (X, d) is a b-metric space with coefficient $K = 2$.

(ii) Let $X = 1, 2, 3$. Define a mapping $d : X \times X \to [0, \infty)$ by $d(1, 1) = d(2, 2) = d(3, 3) = 0, d(1, 2) = d(2, 1) = 2, d(2, 3) = d(3, 2) = 1$ and $d(1, 3) = d(3, 1) = 6$. Then (X, d) is a b-matric space with coefficient $K = 2$.

(iii) The set of real numbers together with the functional

$$d(x, y) := |x - y|^2$$

for all $(x, y \in \varepsilon)$ is a b-metric space with constant $s = 2$. Also, we obtain that d is not a metric on X.

(iv) Let $X = \{0, 1, 2\}$ and a functional $d : X \times X \to \mathbb{R}_+$ be defined by
$d(0, 0) = d(1, 1) = d(2, 2) = 0,$
$d(0, 1) = d(1, 0) = d(1, 2) = d(2, 1) = 1$
and
$d(2, 0) = d(0, 2) = m,$
where m is a given real number such that $m \geq 2$. It is easy to see that

$$d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)]$$

for all $x, y, z \in \varepsilon$. Therefore, (X, d) is a b-metric space with constant $s = \frac{m}{2}$. However, if $m > 2$, the ordinary triangle inequality does not hold and thus (X, d) is not a metric space.

In 2017, Mongkolkeha and et al. [17] introduced a simulation function in the framework of b-metric spaces shown below:

Definition 1.2. *[17] Let K be a given real number such that $K \geq 1$. A K-simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:*
(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(Kt, s) \leq s - Kt$, *for all* $t, s > 0$
(ζ_3) *if* $\{t_n\}, \{s_n\}$ *are sequences in* $[0, \infty)$ *such that* $\lim_{n\to\infty} Kt_n = \lim_{n\to\infty} s_n > 0$ *and* $t_n < s_n$ *for all* $n \in \mathbb{N}$, *then*

$$\lim_{n\to\infty} \zeta(Kt_n, s_n) < 0.$$

The class of all K-simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is denoted by Z^*

Example 1.2. [17] Let $\lambda, K \in \mathbb{R}$ be such that $\lambda < 1$ and $K \geq 1$. Define the mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(Kt, s) = \begin{cases} s - Kt & \text{if } s < t, \\ \frac{\lambda s - Kt}{Ks + 1} & \text{if } otherwise. \end{cases}$$

Then $\zeta \in Z^*$ but $\zeta \notin Z$.

In this year, Wiriyapongsanon and Phudolsitthiphat [18] defined a generalization of R-contraction in b-metric spaces, called R'-contractions, via R'-functions and proved the existence and uniqueness of fixed point for such classes of mappings in complete b-metric spaces.

Definition 1.3. *[18] Let K be a given real number such that $K \geq 1$. A function $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called R'-function if it satisfies the following two conditions:*

(ϱ'_1) If $\{a_n\} \subset (0,\infty)$ is a sequence such that $\varrho(Ka_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

(ϱ'_2) If $\{a_n\}, \{b_n\} \subset (0,\infty)$ are two sequences such that $\limsup\limits_{n\to\infty} Ka_n = \limsup\limits_{n\to\infty} b_n = L \geq 0$ and verifying

that $L < Ka_n$ and $\varrho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then L = 0.

The class of all R'-functions $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$. is denoted by R^* We also consider the following property.

(ϱ'_3) If $\{a_n\}, \{b_n\} \subset (0,\infty)$ are two sequences such that $b_n \to 0$ and $\varrho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then

$a_n \to 0$.

Lemma 1.4. *[18] Every K-simulation function is a R-function that also verifies (ϱ'_3).*

Definition 1.5. *[5] Let (X,d) be a metric space. A mapping $T : X \to X$ is called R-contraction if there exists an R-function $\varrho : A \times A \to \mathbb{R}$ such that $ran(d) \subseteq A$ and*

$$\varrho(d(Tx,Ty), d(x,y)) > 0 \textit{ for all } x, y \in X \textit{ such that } x \neq y.$$

Notice that if we take $\varrho(t,s) = \lambda s - t$ for all $s, t \geq 0$ and $\lambda \in [0,1)$ in Definition 1.5, then R-contraction become the Banach contraction.

Theorem 1.3. *[18] Let (X,d) be a complete b-metric space with coefficient $K \geq 1$. Let $T : X \to X$ be R'-contraction with respect $\varrho \in R^*$. If $\varrho(Kt,s) \leq s - Kt$ for all $s,t \in (0,\infty)$ then T has a unique fixed point.*

In this paper, we extend and improve R'-contractions and via R'-functions mappings to R'-Max-kanan and R''-kanan mappings by using the concept of kanan mappings. Second, we establish new mapping, that is R'-Max-kanan and R''-kanan mappings and prove the results of fixed point for R'-Max-kanan and R''-kanan mappings in b-metric spaces. Moreover, we obtain fixed point theorems for R'-Max-kanan and R''-kanan mappings in b-metric spaces and present some examples to illustrate and support our results which generalized the concept of Kannan [2], Bianchini [3].

2 Results

In this section, we proof fixed point for R'-Max kanan mapping in b-metric spaces.

Theorem 2.1. *Let (X,d) be a complete b-metric space with coefficient $K \geq 1$. Let $T : X \to X$ be R'-Max-kanan mapping , i.e.,*

$$\varrho(2Kd(Tx,Ty), \max\{d(x,Tx), d(y,Ty)\}) > 0$$

with respect $\varrho \in R^*$. If $\varrho(2Kt,s) \leq s - 2Kt$ for all $s,t \in (0,\infty)$ then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a arbitrary point. Let $\{x_n\}$ be Picard sequence of T based on x_0, that is, $x_{n+1} = Tx_n$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_0$, then $Tx_{n_0} = x_n$ Which implies that x_{n_0} is a fixed point. Assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Let $\{a_n\} \subset (0,\infty)$ be a sequence defined by $a_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By R'-Max-kanan mapping,

$$\begin{aligned}
\varrho(2Ka_{n+1}, a_n) &= \varrho(2Kd(x_{n+1}, x_{n+2}), \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\
&= \varrho(2Kd(Tx_n, Tx_{n+1}), \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\}) \\
&> 0.
\end{aligned}$$

If $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$, then

$$\begin{aligned}
0 &< \varrho(2Kd(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) \\
&< d(x_{n+1}, x_{n+2}) - 2Kd(x_{n+1}, x_{n+2}) \\
&= a_n - 2Ka_n \\
&< 0,
\end{aligned}$$

which is a contradiction. Thus

$$\begin{aligned}\varrho(2Ka_{n+1}, a_n + a_{n+1}) &= \varrho(2Kd(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}))\\ &= \varrho(2Kd(Tx_n, Tx_{n+1}), d(x_n, Tx_n))\\ &> 0.\end{aligned}$$

By using the condition (ϱ_1'), we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} a_n = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon_0 > 0$ such that

$$d(x_{n_k}, x_{m_k}) > \varepsilon_0 \text{ and } d(x_{n_k}, x_{m_{k-1}}) < \varepsilon_0 \text{ for all } m_k > n_k \geq k. \tag{2.1}$$

We consider, for any $m_k > n_k \geq k$,

$$\varepsilon_0 < d(x_{n_k}, x_{m_k}) \leq K(d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})) < K(\varepsilon_0 + d(x_{m_{k-1}}, x_{m_k})).$$

Taking limit superior from k to infinity, we have

$$\varepsilon_0 \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq K\varepsilon_0. \tag{2.2}$$

Since $d(x_{n_{k-1}}, x_{m_{k-1}}) \leq K(d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_{k-1}}))$, taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_{k-1}}, x_{m_{k-1}}) \leq K\varepsilon_0. \tag{2.3}$$

If $d(x_{n_{k_0-1}}, x_{m_{k_0-1}}) = 0$ for some $k_0 \in \mathbb{N}$, then $x_{n_{k_0}} = x_{m_{k_0}}$, which contradict to (2.1) Therefore $x_{n_{k-1}} \neq x_{m_{k-1}}$ for all $k_0 \in \mathbb{N}$. By R'-Max-kanan mapping, we get

$$\begin{aligned}0 &< \varrho(2Kd(x_{n_k}, x_{m_k}), \max\{d(x_{n_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k})\})\\ &\leq \max\{d(x_{n_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k})\} - 2Kd(x_{n_k}, x_{m_k}).\end{aligned}$$

So, we have

$$2Kd(x_{n_k}, x_{m_k}) < \max\{d(x_{n_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k})\} \text{ for all } k_0 \in \mathbb{N}. \tag{2.4}$$

By (2.2), (2.3) and (2.4), we get that

$$2K\varepsilon_0 \leq \limsup_{k\to\infty} 2Kd(x_{n_k}, x_{m_k}) \leq \limsup_{k\to\infty}(\max\{d(x_{n_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k})\}).$$

By continuity of maximum functions, we have

$$2K\varepsilon_0 \leq \limsup_{k\to\infty} 2Kd(x_{n_k}, x_{m_k}) = \max\{\limsup_{k\to\infty} d(x_{n_{k-1}}, x_{n_k}), \limsup_{k\to\infty} d(x_{m_{k-1}}, x_{m_k})\} = 0.$$

Since $2K\varepsilon_0 \leq 0$, we have $K\varepsilon_0 = 0$. That is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. By definition of convergence sequence,

$$\text{for any } \varepsilon > 0 \text{ there exists } \mathbb{N} \text{ such that } d(x_n, z) < \varepsilon \text{ for all } n > N. \tag{2.5}$$

Next, we will show that z is fixed point. Let $\Omega = \{n \in \mathbb{N} : d(x_n, z) = 0\}$. Assume that Ω is not finite, then we can find $n_0 > N$ such that $d(x_{n_0}, z) = 0$ i.e. $x_{n_0} = z$. Since $x_{n_0} \neq x_{n_0+1}$ and $x_{n_0+1} = Tx_{n_0} = Tz$, and then $z \neq Tz$.
Let $\varepsilon = \frac{d(z,Tz)}{2} > 0$. By (2.5), we have

$$\varepsilon > d(x_{n_0+1}, z) = d(Tx_{n_0}, z) = d(Tz, z) = 2\varepsilon,$$

which is a contradiction. Therefore Ω is finite, there exists n_0 such that $d(x_n, z) > 0$ for all $n > n_0$. Since T is a R'-Max-kanan mapping,

$$\begin{aligned}0 &< \varrho(2Kd(Tx_n, Tz), \max\{d(x_n, Tx_n), d(z, Tz)\})\\ &\leq \max\{d(x_n, Tx_n), d(z, Tz)\} - 2Kd(Tx_n, Tz).\end{aligned}$$

Hence,

$$2Kd(Tx_n, Tz) \leq \max\{d(x_n, x_{n+1}), d(z, Tz)\}.$$

If $\max\{d(x_n, x_{n+1}), d(z, Tz)\} = d(z, Tz)$, then

$$\begin{aligned} 2Kd(Tx_n, Tz) &\leq d(z, Tz). \\ &\leq Kd(z, Tx_n) + Kd(Tx_n + Tz), \end{aligned}$$

so,

$$Kd(Tx_n, Tz) \leq Kd(z, Tx_n). \tag{2.6}$$

If $\max\{d(x_n, x_{n+1}), d(z, Tz)\} = d(x_n, x_{n+1})$, then

$$Kd(Tx_n, Tz) \leq 2Kd(Tx_n, Tz) \leq d(x_n, x_{n+1}). \tag{2.7}$$

Taking limit n to infinity on (2.6) and (2.7), thus

$$\lim_{n\to\infty} Kd(Tx_n, Tz) = 0.$$

That is $\{x_{n+1} = Tx_n\} \to Tz$. By the uniqueness of the limit, $Tz = z$. Finally, we show that z is unique fixed point of T. Assume $x = Tx$ and $y = Ty$ such that $x \neq y$. We consider

$$\begin{aligned} 0 &< \varrho(2kd(Tx, Ty), \max\{d(x, Tx), d(y, Ty)\}) \\ &< 0 - 2kd(x, y). \end{aligned}$$

So $x = y$. □

Example 2.2. Let $X = [0, 1]$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$, then (X, d) is a complete b-metric space with coefficient $K = 2$. Let $T : X \to X$ be given by $T(x) = \frac{x^2}{\sqrt{11}(3+x)}$ for all $x \in X$. For all $x, y \in X$ such that $x \geq y$, we have

$$\begin{aligned} d(Tx, Ty) &= \left(\frac{x^2}{\sqrt{11}(3+x)} - \frac{y^2}{\sqrt{11}(3+y)} \right)^2 \\ &= \left(\frac{3(x^2 - y^2) + xy(x - y)}{\sqrt{11}(3+x)(3+y)} \right)^2 \\ &= \frac{1}{11} \left(\frac{3(x^2 - y^2) + xy(x - y)}{(3+x)(3+y)} \right)^2 \end{aligned}$$

and we get,

$$\begin{aligned}
\max\{d(x,Tx),d(y,Ty)\} &= \max\left\{\left(x-\frac{x^2}{\sqrt{11}(3+x)}\right)^2,\left(x-\frac{y^2}{\sqrt{11}(3+y)}\right)^2\right\}\\
&= \max\left\{\left(\frac{\sqrt{11}(3x^2+x^2y+3xy+9x)-3x^2-x^2y}{\sqrt{11}(3+x)(3+y)}\right)^2,\right.\\
&\qquad\left.\left(\frac{\sqrt{11}(3y^2+xy^2+3xy+9y)-3y^2-xy^2}{\sqrt{11}(3+x)(3+y)}\right)^2\right\}\\
&= \max\left\{\frac{1}{11}\left(\frac{\sqrt{11}(3x^2+x^2y+3xy+9x)-3x^2-x^2y}{(3+x)(3+y)}\right)^2,\right.\\
&\qquad\left.\frac{1}{11}\left(\frac{\sqrt{11}(3y^2+xy^2+3xy+9y)-3y^2-xy^2}{(3+x)(3+y)}\right)^2\right\}\\
&= \frac{1}{11}\left(\frac{\sqrt{11}(3x^2+x^2y+3xy+9x)-3x^2-x^2y}{(3+x)(3+y)}\right)^2 ;x\geq y\\
&\geq \frac{1}{11}\left(\frac{\sqrt{9}(3x^2+x^2y+3xy+9x)-3x^2-x^2y}{(3+x)(3+y)}\right)^2\\
&= \frac{1}{11}\left(\frac{3(2x^2+9x)+xy(2x+9)}{(3+x)(3+y)}\right)^2.
\end{aligned}$$

Define $\varrho : [0,\infty)\times[0,\infty)\to\mathbb{R}$ by $\varrho(2t,s)=\frac{s}{1+s}-2t$, then $\varrho\in R^*$.
Therefore

$$\begin{aligned}
&\varrho(2d(Tx,Ty),\max\{d(x,Tx),d(y,Ty)\})\\
&\geq \frac{\frac{1}{11}\left(\frac{3(2x^2+9x)+xy(2x+9)}{(3+x)(3+y)}\right)^2}{1+\frac{1}{11}\left(\frac{3(2x^2+9x)+xy(2x+9)}{(3+x)(3+y)}\right)^2}-\frac{2}{11}\left(\frac{3(x^2-y^2)+xy(x-y)}{(3+x)(3+y)}\right)^2\\
&\geq \frac{\frac{1}{11}\left(\frac{3(2x^2+9x)+xy(2x+9)}{(3+x)(3+y)}\right)^2}{1+(\frac{1}{11})(\frac{44^2}{9^2})}-\frac{2}{11}\left(\frac{3(x^2-y^2)+xy(x-y)}{(3+x)(3+y)}\right)^2\\
&\geq \frac{1}{44}\left(\frac{3(2x^2+9x)+xy(2x+9)}{(3+x)(3+y)}\right)^2-\frac{2}{11}\left(\frac{3(x^2-y^2)+xy(x-y)}{(3+x)(3+y)}\right)^2\\
&\geq \frac{1}{44}\left(\frac{36x^4+324x^3+729x^2+24x^4y+216x^3y+486x^2y+4x^4y^2+36x^3y^2+81x^2y^2}{(3+x)^2(3+y)^2}\right)\\
&\quad-\frac{2}{11}\left(\frac{9x^4-18x^2y^2+9y^4+6x^4y-6x^3y^2-6x^2y^3+6xy^4+x^4y^2-2x^3y^3+x^2y^4}{(3+x)^2(3+y)^2}\right)\\
&\geq \frac{1}{44}\left(\frac{\begin{array}{c}-36x^4+324x^3+729x^2-24x^4y+216x^3y+486x^2y-4x^4y^2+\\ 84x^3y^2+225x^2y^2-72y^4+48x^2y^3-48xy^4+16x^3y^3-8x^2y^4\end{array}}{(3+x)^2(3+y)^2}\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{44}\left(\frac{-36x^4+324x^3+729x^2-24x^4y+212x^2y^2+486x^2y+84x^3y^2+153y^4+8x^2y^4}{(3+x)^2(3+y)^2}\right) \quad ; x \geq y\\
&\geq \frac{1}{44}\left(\frac{288x^4+729x^2-24x^4y+212x^2y^2+486x^2y+84x^3y^2+153y^4+8x^2y^4}{(3+x)^2(3+y)^2}\right) \quad ; x^3 \geq x^4\\
&\geq \frac{1}{44}\left(\frac{264x^5+729x^2+212x^2y^2+486x^2y+84x^3y^2+153y^4+8x^2y^4}{(3+x)^2(3+y)^2}\right) \quad ; x^4 \geq x^5\\
&\geq 0.
\end{aligned}$$

Thus T satisfies the R′-Max-kanan mapping of Theorem 2.1 and 0 is the unique fixed point of T.

Definition 2.1. *Let K be a given real number such that $K \geq 1$. A function $\varrho : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called R''-function if it satisfies the following two conditions:*

(ϱ_1') *If $\{a_n\} \subset (0,\infty)$ is a sequence such that $\varrho(2Ka_{n+1}, a_n + a_{n+1}) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.*

(ϱ_2') *If $\{a_n\},\{b_n\} \subset (0,\infty)$ are two sequences such that $\limsup_{n\to\infty} Ka_n = \limsup_{n\to\infty} b_n = L \geq 0$ and verifying that*

$L < Ka_n$ and $\varrho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

The class of all R''-functions $\varrho : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is denoted by R^{**}. We also consider the following property.

(ϱ_3') *If $\{a_n\},\{b_n\} \subset (0,\infty)$ are two sequences such that $b_n \to 0$ and $\varrho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.*

Theorem 2.3. *Let (X,d) be a complete b-metric space with coefficient $K \geq 1$. Let $T : X \to X$ be R''-kanan mapping , i.e.,*

$$\varrho(2Kd(Tx,Ty), d(x,Tx) + d(y,Ty)) > 0$$

with respect $\varrho \in R^{**}$. If $\varrho(2Kt,s) \leq s - 2Kt$, for all $s,t \in (0,\infty)$ then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a arbitrary point. Let $\{x_n\}$ be Picard sequence of T based on x_0, that is, $x_{n+1} = Tx_n$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_0$, then $Tx_{n_0} = x_n$ Which implies that x_{n_0} is a fixed point. Assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Let $\{a_n\} \subset (0,\infty)$ be a sequence defined by $a_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By R''-kanan mapping,

$$\begin{aligned}
\varrho(2Ka_{n+1}, a_n + a_{n+1}) &= \varrho(2Kd(x_{n+1},x_{n+2}), d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}))\\
&= \varrho(2Kd(Tx_n,Tx_{n+1}), d(x_n,Tx_n) + d(x_{n+1},Tx_{n+1}))\\
&> 0.
\end{aligned}$$

By using the condition (ϱ_1'), we get that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \lim_{n\to\infty} a_n = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon_0 > 0$ such that

$$d(x_{n_k}, x_{m_k}) > \varepsilon_0 \ \text{ and } \ d(x_{n_k}, x_{m_{k-1}}) < \varepsilon_0 \text{ for all } \ m_k > n_k \geq k. \tag{2.8}$$

We consider, for any $m_k > n_k \geq k$,

$$\varepsilon_0 < d(x_{n_k}, x_{m_k}) \leq K(d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})).$$

Taking limit superior form k to infinity, we have

$$\varepsilon_0 \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq K\varepsilon_0. \tag{2.9}$$

Since $d(x_{n_{k-1}}, x_{m_{k-1}}) \leq K(d(x_{n_{k-1},x_{n_k}}) + d(x_{n_k}, x_{m_{k-1}}))$, taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_{k-1}}, x_{m_{k-1}}) \leq K\varepsilon_0. \tag{2.10}$$

If $d(x_{n_{k_0-1}}, x_{m_{k_0-1}}) = 0$ for some $k_0 \in \mathbb{N}$ then $x_{n_{k_0}} = x_{m_{k_0}}$, which contradict to (2.8) Therefore $x_{n_{k-1}} \neq x_{m_{k-1}}$ for all $k_0 \in \mathbb{N}$. By R''-kanan mapping,

$$\begin{aligned}
0 &< \varrho(2Kd(x_{n_k}, x_{m_k}), d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k}))\\
&\leq [d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k})] - 2Kd(x_{n_k}, x_{m_k}).
\end{aligned}$$

So, we have

$$2Kd(x_{n_k}, x_{m_k}) < d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k}) \text{ for all } k_0 \in \mathbb{N}. \tag{2.11}$$

By (2.9), (2.10) and (2.11), we get that

$$K\varepsilon_0 \leq \limsup_{k\to\infty} 2Kd(x_{n_k}, x_{m_k}) \leq \limsup_{k\to\infty}[d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k})] \leq K\varepsilon_0.$$

Thus

$$\limsup_{k\to\infty} 2Kd(x_{n_k}, x_{m_k}) = \limsup_{k\to\infty}[d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k})] = K\varepsilon_0.$$

Since $K\varepsilon_0 < 2Kd(x_{n_k}, x_{m_k})$, for all $k_0 \in \mathbb{N}$ and the condition (ϱ_2'), $K\varepsilon_0 = 0$. This is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. By definition of convergence sequence,

$$\text{for any}\varepsilon > 0\text{there exists}\mathbb{N}\text{such that}d(x_n, z) < \varepsilon\text{for all}n > N. \tag{2.12}$$

Next, we will show that z is fixed point. Let $\Omega = \{n \in \mathbb{N} : d(x_n, z) = 0\}$. Assume that Ω is not finite, then we can find $n_0 > N$ such that $d(x_{n_0}, z) = 0$ i.e. $x_{n_0} = z$. Since $x_{n_0} \neq x_{n_0+1}$ and $x_{n_0+1} = Tx_{n_0} = Tx$, and then $z \neq Tz$. Let $\varepsilon = \frac{d(z,Tz)}{2} > 0$. By (2.12),we have

$$\varepsilon > d(x_{n_0+1}, z) = d(Tx_{n_0}, z) = d(Tz, z),$$

which is a contradiction. Therefore Ω is finite, there exists n_0 such that $d(x_n, z) > 0$ for all $n > n_0$. Since T is a R''-Kanan mapping,

$$0 < \varrho(2Kd(Tx_n, Tz), d(x_n, Tx_n) + d(z, Tz)) \leq [d(x_n, Tx_n) + d(z, Tz)] - 2Kd(Tx_n, Tz).$$

Hence,

$$\begin{aligned} 2Kd(Tx_n, Tz) &\leq d(x_n, Tx_n) + d(z, Tz) \\ &\leq d(x_n, Tx_n) + k[d(z, Tx_n) + d(Tx_n, Tz)] \\ Kd(Tx_n, Tz) &\leq d(x_n, x_{n+1}) + Kd(z, x_{n+1}). \end{aligned}$$

Taking limit n to infinity,

$$\limsup_{n\to\infty} Kd(Tx_n, Tz) = \limsup_{n\to\infty}[d(x_n, x_{n+1}) + Kd(z, x_{n+1})] = 0.$$

Thus

$$\lim_{n\to\infty} Kd(Tx_n, Tz) = 0.$$

That is $\{x_{n+1} = Tx_n\} \to Tz$. By the uniqueness of the limit, $Tz = z$. Finally, let us show that z is unique fixed point of T. Assume $x = Tx$ and $y = Ty$ such that $x \neq y$. Let $a_n = d(x, y) > 0$ for all $n \in \mathbb{N}$. We consider

$$\begin{aligned} \varrho(2Ka_{n+1}, a_n + a_{n+1}) &= \varrho(2Kd(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})) \\ &= \varrho(2Kd(Tx_n, Tx_{n+1}), d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})) \\ &> 0. \end{aligned}$$

By using (ϱ_1'), we get $a_n \to 0$, which imply that $d(x, y) = 0$, which is a contradiction. So $x = y$. □

Example 2.4. Let $X = [0, 1]$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$, then (X, d) is a complete b-metric space with coefficient $K = 2$. Let $T : X \to X$ be given by $T(x) = \frac{x^2}{\sqrt{5}(4+x)}$ for all $x \in X$. For all $x, y \in X$ such that $x \geq y$, we have

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x^2}{\sqrt{5}(4+x)} - \frac{y^2}{\sqrt{5}(4+y)} \right|^2 \\ &= \left| \frac{x^2(4+y) - y^2(4+x)}{\sqrt{5}(4+x)(4+y)} \right|^2 \\ &= \frac{1}{5} \left| \frac{4x^2 + x^2y - 4y^2 - xy^2}{(4+x)(4+y)} \right|^2 \end{aligned}$$

and we have,

$$
\begin{aligned}
d(x,Tx)+d(y,Ty) &= \left|x-\frac{x^2}{\sqrt{5}(4+x)}\right|^2+\left|y-\frac{y^2}{\sqrt{5}(4+y)}\right|^2 \\
&= \left|\frac{\sqrt{5}(4x+x^2)(4+y))-x(4+y)}{\sqrt{5}(4+x)(4+y)}\right|^2+ \\
&\qquad \left|\frac{\sqrt{5}(4y+y^2)(4+x))-y(4+x)}{\sqrt{5}(4+x)(4+y)}\right|^2 \\
&= \frac{1}{5}\left(\left|\frac{\sqrt{5}(16x+4xy+4x^2+x^2y)-4x-4y}{(4+x)(4+y)}\right|^2+\right. \\
&\qquad \left.\left|\frac{\sqrt{5}(16y+4xy+4y^2+xy^2)-4y-4x}{(4+x)(4+y)}\right|^2\right) \\
&\geq \frac{1}{5}\left(\left|\frac{\sqrt{4}(16x+4xy+4x^2+x^2y)-4x-4y}{(4+x)(4+y)}\right|^2+\right. \\
&\qquad \left.\left|\frac{\sqrt{4}(16y+4xy+4y^2+xy^2)-4y-4x}{(4+x)(4+y)}\right|^2\right) \\
&= \frac{1}{5}\left(\left|\frac{28x+8xy+8x^2+2x^2y-4y}{(4+x)(4+y)}\right|^2+\left|\frac{28y+8xy+8y^2+2xy^2-4x}{(4+x)(4+y)}\right|^2\right)
\end{aligned}
$$

Define $\varrho:[0,\infty)\times[0,\infty)\to\mathbb{R}$ by $\varrho(2t,s)=\frac{s}{1+s}-2t$, then $\varrho\in R^{**}$.
Therefore

$$
\begin{aligned}
&\varrho(2d(Tx,Ty),d(x,Tx)+d(y,Ty)) \\
&\geq \frac{\frac{1}{5}\left(\left|\frac{28x+8xy+8x^2+2x^2y-4y}{(4+x)(4+y)}\right|^2+\left|\frac{28y+8xy+8y^2+2xy^2-4x}{(4+x)(4+y)}\right|^2\right)}{1+\frac{1}{5}\left(\left|\frac{28x+8xy+8x^2+2x^2y-4y}{(4+x)(4+y)}\right|^2+\left|\frac{28y+8xy+8y^2+2xy^2-4x}{(4+x)(4+y)}\right|^2\right)}-\frac{2}{5}\left|\frac{4x^2+x^2y-4y^2-xy^2}{(4+x)(4+y)}\right|^2 \\
&\geq \frac{\frac{1}{5}\left(\left|\frac{28x+8xy+8x^2+2x^2y}{(4+x)(4+y)}\right|^2+\left|\frac{28y+8xy+8y^2+2xy^2-4x}{(4+x)(4+y)}\right|^2\right)}{1+\frac{1}{5}\left(\frac{529}{320}\right)}-\frac{2}{5}\left|\frac{4x^2+x^2y-4y^2-xy^2}{(4+x)(4+y)}\right|^2 \\
&\geq \frac{320}{2,129}\left(\left|\frac{28x+8xy+8x^2+2x^2y-4y}{(4+x)(4+y)}\right|^2+\left|\frac{28y+8xy+8y^2+2xy^2-4x}{(4+x)(4+y)}\right|^2\right)-\frac{2}{5}\left|\frac{4x^2+x^2y-4y^2-xy^2}{(4+x)(4+y)}\right|^2 \\
&= \frac{1,280}{2,129}\left(\left|\frac{14x+4xy+4x^2+x^2y-2y}{(4+x)(4+y)}\right|^2+\left|\frac{14y+4xy+4y^2+xy^2-2x}{(4+x)(4+y)}\right|^2\right)-\frac{2}{5}\left|\frac{4x^2+x^2y-4y^2-xy^2}{(4+x)(4+y)}\right|^2 \\
&\geq \frac{1,280}{2,129}\left|\frac{12x+4xy+4x^2+x^2y}{(4+x)(4+y)}\right|^2-\frac{2}{5}\left|\frac{4x^2+x^2y}{(4+x)(4+y)}\right|^2 ;x\geq y \\
&\geq 0
\end{aligned}
$$

Thus T satisfies the R''-kanan mapping of Theorem 2.3 and 0 is the unique fixed point of T.

3 Discussion

Future research directions may also be possible.

Open problems 1:
If T satisfies

$$\varrho(3Kd(Tx,Ty),\max\{d(x,y),d(x,Tx),d(y,Ty)\})>0$$

then T, has a unique fixed point.

Open problems 2:
If T,S satisfies

$$\varrho(3Kd(Tx,Sy),d(x,y)+d(x,Tx)+d(y,Sy))>0$$

then T,S has a unique common fixed point.

4 Conclusions

In this paper, we extend and improve R'-contractions and via R'-functions mappings to R'-Max-kanan and R''-kanan mappings by using the concept of kanan mappings. Second, we establish new mapping, that is R'-Max-kanan and R''-kanan mappings and prove the results of fixed point for R'-Max-kanan and R''-kanan mappings in b-metric spaces. Moreover, we obtain fixed point theorems for R'-Max-kanan and R''-kanan mappings in b-metric spaces and present some examples to illustrate and support our results as follows:

1. A b-metric on a set X is a mapping $d: X\times X\to[0,+\infty)$ satisfying the following conditions: for any $x,y,z\in X$,
(b_1) $d(x,y)=0$ if and only if $x=y$;
(b_2) $d(x,y)=d(y,x)$;
(b_3) there exists $K\geq 1$ such that $d(x,y)\leq K(d(x,z)+d(z,y))$.
Then (X,d) is known as a b-metric space with coefficient K.

2. Let (X,d) be a complete b-metric space with coefficient $K\geq 1$. Let $T: X\to X$ be R'-Max-kanan mapping , i.e.,

$$\varrho(2Kd(Tx,Ty),\max\{d(x,Tx),d(y,Ty)\})>0$$

with respect $\varrho\in R^*$. If $\varrho(2Kt,s)\leq s-2Kt$ for all $s,t\in(0,\infty)$ then T has a unique fixed point.

3. Let $X=[0,1]$ and $d(x,y)=|x-y|^2$ for all $x,y\in X$, then (X,d) is a complete b-metric space with coefficient $K=2$. Let $T: X\to X$ be given by $T(x)=\frac{x^2}{\sqrt{11}(3+x)}$ for all $x\in X$.

4. Let K be a given real number such that $K\geq 1$. A function $\varrho:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called R''-function if it satisfies the following two conditions:
(ϱ_1') If $\{a_n\}\subset(0,\infty)$ is a sequence such that $\varrho(2Ka_{n+1},a_n+a_{n+1})>0$ for all $n\in\mathbb{N}$, then $a_n\to 0$.
(ϱ_2') If $\{a_n\},\{b_n\}\subset(0,\infty)$ are two sequences such that $\limsup\limits_{n\to\infty}Ka_n=\limsup\limits_{n\to\infty}b_n=L\geq 0$ and verifying
that $L<Ka_n$ and $\varrho(Ka_n,b_n)>0$ for all $n\in\mathbb{N}$, then = 0.
The class of all R''-functions $\varrho:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is denoted by R^{**}. We also consider the following property.
(ϱ_3') If $\{a_n\},\{b_n\}\subset(0,\infty)$ are two sequences such that $b_n\to 0$ and $\varrho(Ka_n,b_n)>0$ for all $n\in\mathbb{N}$, then
$a_n\to 0$.

5. Let (X,d) be a complete b-metric space with coefficient $K\geq 1$. Let $T: X\to X$ be R''-kanan mapping , i.e.,

$$\varrho(2Kd(Tx,Ty),d(x,Tx)+d(y,Ty))>0$$

with respect $\varrho\in R^{**}$. If $\varrho(2Kt,s)\leq s-2Kt$, for all $s,t\in(0,\infty)$ then T has a unique fixed point.

Acknowledgments: The authors would like to thank the Research and Development Institute of Kamphaeng Phet Rajabhat University, which provides funding for research.

References

[1] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund Math. 1922;3:133–181.

[2] Kannan R. Some results on fixed points-II. Amer Math Monthly. 1969;76:405–408.

[3] Bianchini RMT. Su un problem a di S. Reich aguardante la teoria dei punti fissi. Boll Un Mat Ital. 1969;5.

[4] Khojasteh F, Shukla S, Radenović S. A new approach to the study of fixed point theorems via simulation functions. Filomat. 2015;96:1189–1194.

[5] Roldán López de Hierro AF, Shahzad N. New fixed point theorem under R-contractions. Fixed Point Theory and Applications. 2015 Jul;2015(1):98. Available from: `https://doi.org/10.1186/s13663-015-0345-y`.

[6] Bakhtin A. The contraction mapping principle in quasimetric spaces. Funct Anal. 1989;30:26–37.

[7] Czerwik S. Contraction mapping in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis. 1993;1:5–11.

[8] Phiangsungnoen S, Kumam P. On stability of fixed point inclusion for multivalued type contraction mappings in dislocated b-metric spaces with application. Mathematical Methods in the Applied Sciences;41(17):7407–7420. Available from: `https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.4871`.

[9] Singh D, Chauhan V, Kumam P, Joshi V. Some applications of fixed point results for generalized two classes of Boyd–Wong's F-contraction in partial b-metric spaces. Mathematical Sciences. 2018 Jun;12(2):111–127. Available from: `https://doi.org/10.1007/s40096-018-0250-8`.

[10] Kumam W, Sukprasert P, Kumam P, Shoaib A, Shahzad A, Mahmood Q. Some fuzzy fixed point results for fuzzy mappings in complete b-metric spaces. Cogent Mathematics & Statistics. 2018;5. Https://doi.org/10.1080/25742558.2018.1458933.

[11] Kumam P, Dung NV, Hang VTL. Some equivalences between cone b-metric spaces and b-metric spaces. Abstract and Applied Analysis. 2013;8.

[12] Phiangsungnoen S, Kumam P. Fuzzy fixed point theorems for multival ued fuzzy contractions in b-metric spaces. J Nonlinear Sci Appl. 2015;8(5563).

[13] Piri H, Kumam P. Fixed point theorems for generalized F-Suzuki contraction in complete b-metric spaces. Fixed Point Theory and Applications. 2016;90. Doi: 10.1186/s13663-016-0577-5.

[14] Klin-eam C, Suanoom C. Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions. Fixed Point Theory and Applications. 2015;p. 1–3.

[15] Klin-eam C, Suanoom C, Suantai S. Dislocated quasi-b-metric spaces and fixed point theorems for cyclic weakly contractions. J Nonlinear Sci Appl. 2016;9:2779–2780.

[16] Sintunavarat W, Plubtieng S, Katchang P. Fixed point result and applications on a b-metric space endowed with an arbitrary binary relation. Fixed Point Theory and Applications. 2013 Nov;2013(1):296. Available from: `https://doi.org/10.1186/1687-1812-2013-296`.

[17] Mongkolkeha C, Cho YJ, Kumam P. Fixed point theorems for simulation functions in b-metric spaces via the wt-distance. Applied General Topology. 2017;18(1).

[18] Wiriyapongsanon A, Phudolsitthiphat N. Fixed Point Theorems for Generalized R-Contraction in b-Metric Spaces. Thai Journal of Mathematics: Special Issue (ACFPTO2018) on Advances in fixed point theory towards real world optimization problems. 2018;.