Some Fixed Point Results for Generalized R'-Contraction and Generalized R'-Kannan Mappings in b-Metric Spaces.

Chatsuda Chanmanee[a,1] , Cholatis Suanoom[a,2] and Patchara Muangkarn[a,3]

[a] Program of Mathematics, Faculty of Science and Technology, Kamphaengphet Rajabhat University, Kamphaengphet, Thailand.
[1]E-mail: chatsuda.chanmanee@gmail.com (Chatsuda Chanmanee)
[2]Correspondence: cholatis.suanoom@gmail.com (Cholatis Suanoom)
[3] E-mail: putchara_31@windowslive.com (Patchara Muangkarn)

Abstract

The purpose of this paper is to extend and improve some results concerning of R'-Max-kannan and R''-kannan mappings to generalize R'-contraction and generalized R'-Kannan mappings. Second, we establish new mapping, that is generalized R'-contraction and generalized R'-Kannan mappings and prove the results of fixed point for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces. Moreover, we obtain fixed point theorems for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces.

Keywords: fixed point / b-metric spaces / R'-Max-kannan mappings / R''-kannan mappings / R-function / K-simulation function.

Introduction and Preliminaries.

The concept of a b-metric space was introduced by Czerwik in $[8]$ as a generalization of a metric space where the triangular inequality is replaced by $d(x,y) \le r[d(x,z)+d(z,y)]$ with $r \ge 1$, for all $x,y,z \in X$. Since then, several papers deal with fixed point theory for single valued and multivalued operators in b-metric spaces, for instance, see $([1]-[4])$ and Bakhtin $[5]$ and Czerwik $[8]$ developed the notion of b-metric space and established some fixed point theorems in b-metric spaces. Subsequently, several results appeared in this direction $([12]-[15],[17]-[19],[21])$.

On the other hand, let (X,d) be a metric space and let T be a mapping of X into itself. A mapping T is a contraction if there exists a number $r, 0 \le r < 1$, such that the condition

$$d(Tx,Ty) \le rd(x,y) \text{ for all } x,y \in X. \qquad (1.1)$$

The well-known Banach contraction principle is the following: If $T: X \to X$ is a *contraction mapping* of a *complete metric space* X into itself, then

1. There is in X a unique fixed-point
2. $T^n x \to x$ for all $x \in X$ and

3. $d(T^n x,x)\le \frac{r^n}{1-r}d(x,Tx).$

The theorem of Banach and its extensions usually are proved by the fact that the geometrical series $\sum_{n=0}^{\infty} r^n$ is convergent. Some different proof of the Banach theorem is given by R.Kannan $[10]$, where he investigated properties of subsets of X, defined as $S_r=\{x\in X : d(x,Tx)\le r\}, 0<r<+\infty.$ Further, in $[9]$ he showed the following: If X is *a complete metric space* and mapping $T: X\to X$ is such that the condition

$$d(Tx,Ty)\le r(d(x,Tx)+d(y,Ty));\ 0<r<\frac{1}{2} \text{ for all } x,y\in X. \tag{1.2}$$

Then T leaves exactly one point of X fixed. The condition (1.1) and (1.2) are independent, as it was shown by two examples in $[9]$ in 1972, Bianchini $[6]$ introduced generalized Kannan mapping which generalized the concept of Kannan $[9]$ as follows: Let T be a self-mapping on metric spaces $X.$ A mapping T is called *a generalized Kannan mapping or Bianchini mapping* if there exists $r\in[0,1)$ such that

$$d(Tx,Ty)\le r\max\{d(x,Tx),d(y,Ty)\}, \text{ for all } x,y\in X. \tag{1.3}$$

In 2015, Khojasteh et al. $[11]$ introduced the notion of Z-contraction defined by simulation function. Then Khojasteh et al. proved a new fixed point theorem concerning Z-contraction which generalizes Banach's contraction principle. Recently, Roldan-López-*de*-*Hierro* and Shahzad $[20]$ introduced the concept of R-contraction defined by R-function in order to generalize the previous results.

Definition 1.1. $[8]$ Let X be a non-empty set. A b-metric on a set X is a mapping $d: X\times X\to[0,+\infty)$ satisfying the following conditions: for any $x,y,z\in X,$

(b_1) $d(x,y)=0$ if and only if $x=y,$

(b_2) $d(x,y)=d(y,x);$

(b_3) there exists $K\ge 1$ such that $d(x,y)\le K(d(x,z)+d(z,y)).$

Then (X,d) is known as a b-metric spaces with coefficient $K.$

Note that every metric spaces is a b-metric space with $K=1.$ Some examples of b-metric spaces are given below:

Example 1.1 $[8]$

(i) Let $X=[0,\infty)$ and $d: X\times X\to \mathbb{R}^+$ is a mapping defined a $d(x,y):=|x-y|^p$. Then (X,d) is a b-metric spaces, where $1<p$ is a real number.

(ii) Let $X=\{0,1,2\}$ and a functional $d: X\times X\to\mathbb{R}^+$ be defined by

$d(0,0) = d(1,1) = d(2,2) = 0,$

$d(0,1)=d(1,0)=d(1,2)=d(2,1)=1$ and

$d(2,0)=d(0,2)=m,$

where m is a given real number such that $m\ge 2.$ It is easy to see that

$$d(x,y) \le \frac{m}{2}[d(x,z)+d(z,y)]$$

for all $x,y,z \in \mathbb{R}$. Therefore, (X,d) is a b-metric space with constant $s=\frac{m}{2}$. However, if $m>2$, the ordinary triangle inequality does not hold and thus (X,d) is not a metric space.

In 2017, Mongkolkeha and et al. [16] introduced a simulation function in the framework of b-metric spaces showed below:

Definition 1.2. [16] Let K be a given real number such that $K \ge 1$. A K-simulation function is a mapping $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0)=0;$

(ζ_2) $\zeta(Kt,s)\le s-Kt,$ for all $t,s>0$

(ζ_3) if $\{t_n\},\{s_n\}$ are sequences in $[0,\infty)$ such that $\lim_{n\to\infty} Kt_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n\in\mathbb{N}$, then

$$\lim_{n\to\infty}\zeta(Kt_n,s_n)<0.$$

The class of all K-simulation functions $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is denoted by Z^*

Example 1.2 [16] Let $\lambda,K\in\mathbb{R}$ be such that $\lambda<1$ and $K\ge1$. Define the mapping $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ by

Then $\zeta\in Z^*$ but $\zeta\notin Z$, where Z is simulation functions and Z^* is K-simulation functions.

In 2018, Wiriyapongsanon and Phudolsitthiphat [22] defined a generalization of R-contraction in b-metric spaces, called R'-contractions, via R'-functions and proved the existence and uniqueness of fixed point for such classes of mappings in complete b-metric spaces.

Definition 1.3. [22] Let K be a given real number such that $K\ge1$. A function $\tilde{n}:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called R'-function if it satisfies the following two conditions:

$(\tilde{n}_1')$ If $\{a_n\}\subset(0,\infty)$ is a sequence such that $\tilde{n}(Ka_{n+1},a_n)>0$ for all $n\in\mathbb{N}$, then $a_n\to0.$

$(\tilde{n}_2')$ If $\{a_n\},\{b_n\}\subset(0,\infty)$ are two sequences such that $\limsup_{n\to\infty} Ka_n = \limsup_{n\to\infty} b_n = L\ge0$ and verifying that $L<Ka_n$ and $\tilde{n}(Ka_n,b_n)>0$ for all $n\in\mathbb{N}$, then $L=0.$

The class of all R'-functions $\tilde{n}:[0,\infty)\times[0,\infty)\to\mathbb{R}$. is denoted by R^* We also consider the following property.

$(\tilde{n}_3')$ If $\{a_n\},\{b_n\}\subset(0,\infty)$ are two sequences such that $b_n\to0$ and $\tilde{n}(Ka_n,b_n)>0$ for all $n\in\mathbb{N}$, then $a_n\to0.$

Lemma 1.1. [22] Every K-simulation function is a R-function that also verifies $(\tilde{n}_3')$.

Definition 1.4. [20] Let (X,d) be a metric space. A mapping $T:X\to X$ is called R-contraction if there exists an R-function $\tilde{n}:A\times A\to\mathbb{R}$ such that $ran(d)\subseteq A$ and

$$\tilde{n}(d(Tx,Ty),d(x,y))>0 \text{ for all } x,y\in X \text{ such that } x\neq y.$$

Notice that if we take $\tilde{n}(t,s)=\lambda s-t$ for all $s,t\geq 0$ and $\lambda\in[0,1)$ in Definition 1.5, then R-contraction become the Banach contraction.

Theorem 1.1. [22] Let (X,d) be a complete b-metric space with coefficient $K\geq 1$. Let $T:X\to X$ be R'-contraction with respect $\tilde{n}\in R^*$. If $\tilde{n}(Kt,s)\leq s-Kt$ for all $s,t\in(0,\infty)$ then T has a unique fixed point.

In this year, Cholatis et al. [7] mappings to R'-Max-kanan and R''-kanan mappings by using the concept of kanan mappings. Second, who establish new mapping, that is R'-Max-kanan and R''-kanan mappings and prove the results of fixed point for R'-Max-kanan and R''-kanan mappings in b-metric spaces. Moreover, who obtain fixed point theorems for R'-Max-kanan and R''-kanan mappings in b-metric spaces

Theorem 1.2. [7] Let (X,d) be a complete b-metric space with coefficient $K\geq 1$. Let $T:X\to X$ be R'-Max-kanan mapping , i.e.,

$$\tilde{n}(2Kd(Tx,Ty),\max\{d(x,Tx),d(y,Ty)\})>0$$

with respect $\tilde{n}\in R^*$. If $\tilde{n}(2Kt,s)\leq s-2Kt$ for all $s,t\in(0,\infty)$ then T has a unique fixed point.

Example 1.3. [7] Let $X=[0,1]$ and $d(x,y)=|x-y|^2$ for all $x,y\in X$, then (X,d) is a complete b-metric space with coefficient $K=2$. Let $T:X\to X$ be given by

$$T(x)=\frac{x^2}{\sqrt{11(3+x)}} \text{ for all } x\in X.$$

Definition 1.5. Let K be a given real number such that $K\geq 1$. A function $\tilde{n}:[0,\infty)\times[0,\infty)\to\square$ is called R''-function if it satisfies the following two conditions:

($\tilde{n}_1'$) If $\{a_n\}\subset(0,\infty)$ is a sequence such that $\tilde{n}(2Ka_{n+1},a_n+a_{n+1})>0$ for all $n\in\square$, then $a_n\to 0$.

($\tilde{n}_2'$) If $\{a_n\},\{b_n\}\subset(0,\infty)$ are two sequences such that $\limsup_{n\to\infty} Ka_n=\limsup_{n\to\infty} b_n=L\geq 0$ and verifying that $L<Ka_n$ and $\tilde{n}(Ka_n,b_n)>0$ for all $n\in\square$, then $L=0$.

The class of all R'-functions $\tilde{n}:[0,\infty)\times[0,\infty)\to\square$. is denoted by R^* We also consider the following property.

($\tilde{n}_3'$) If $\{a_n\},\{b_n\}\subset(0,\infty)$ are two sequences such that $b_n\to 0$ and $\tilde{n}(Ka_n,b_n)>0$ for all $n\in\square$, then $a_n\to 0$.

Theorem 1.3. [7] Let (X,d) be a complete b-metric space with coefficient $K\geq 1$. Let $T:X\to X$ be R''-kannan mapping , i.e.,

$$\tilde{n}(2Kd(Tx,Ty),d(x,Tx)+d(y,Ty))>0$$

with respect $\tilde{n}\in R^{**}$. If $\tilde{n}(2Kt,s)\leq s-2Kt,$ for all $s,t\in(0,\infty)$ then T has a unique fixed point.

Example 1.4. [7] Let $X=[0,1]$ and $d(x,y)=|x-y|^2$ for all $x,y\in X$, then (X,d) is a complete b-metric space with coefficient $K=2.$ Let $T:X\to X$ be given by

$$T(x)=\frac{x^2}{\sqrt{5}(4+x)} \text{ for all } x\in X.$$

The purpose of this paper is to extend and improve some results concerning of R'-Max-kannan and R''-kannan mappings to generalized R'-contraction and generalized R'-Kannan mappings. Second, we establish new mapping, that is generalized R'-contraction and generalized R'-Kannan mappings and prove the results of fixed point for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces. Moreover, we obtain fixed point theorems for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces which generalized the concept of Kannan [9], Bianchini [6] and Wiriyapongsanon and Phudolsitthiphat [22].

Research objectives

1. To extend and improve some results concerning of R'-Max-kannan and R''-kannan mappings to generalized R'-contraction and generalized R'-Kannan mappings.
2. To establish new mapping, that is generalized R'-contraction and generalized R'-Kannan mappings and prove the results of fixed point for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces.
3. To obtain fixed point theorems for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces.

Results

In this section, we prove fixed point theorems for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces.

Definition 2.1. Let K be a given real number such that $K\geq 1.$ A function $ñ:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called **generalized R-function** if it satisfies the following two conditions:

$(ñ_1)$ If $\{a_n\}\subset(0,\infty)$ is a sequence such that $ñ(2Ka_{n+1},2a_n+a_{n+1})>0$ for all $n\in\mathbb{N}$, then $a_n\to 0.$

$(ñ_2)$ If $\{a_n\},\{b_n\}\subset(0,\infty)$ are two sequences such that $\limsup\limits_{n\to\infty} Ka_n=\limsup\limits_{n\to\infty} b_n=L\geq 0$ and verifying that $L<Ka_n$ and $ñ(Ka_n,b_n)>0$ for all $n\in\mathbb{N}$, then $L=0.$

Theorem 2.1. Let (X,d) be a complete b-metric space with coefficient $K\geq 1,$ and suppose that let $T:X\to X$ be generalized R'-contraction mapping, i.e.

$$ñ(2Kd(Tx,Ty),d(x,y)+d(x,Tx)+d(y,Ty))>0$$

with respect $\tilde{n} \in R^*$. If $\tilde{n}(2Kt,s) \le s - 2Kt$ for all $s,t \in (0,\infty)$ then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a arbitrary point. Let $\{x_n\}$ be Picard sequence of T based on x_0, that is $x_{n+1} = Tx_n$ for all $n \ge 1$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ which implies that x_{n_0} is a fixed point. Assume $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Let $\{a_n\} \subset (0,\infty)$ be a sequence defined by $a_n = d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By generalized R'-contraction mapping,

$$\begin{aligned} \tilde{n}(2Ka_{n+1}, 2a_n + a_{n+1}) &= \tilde{n}(2Kd(x_{n+1},x_{n+2}), d(x_n,x_{n+1}) + d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})) \\ &= \tilde{n}(2Kd(Tx_n,Tx_{n+1}), d(x_n,x_{n+1}) + d(x_n,Tx_n) + d(x_{n+1},Tx_{n+1})) \\ &> 0. \end{aligned}$$

By using the condition $(\tilde{n}_1)$, we get that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = \lim_{n\to\infty} a_n = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon_0 > 0$ such that

$$d(x_{n_k}, x_{m_k}) > \varepsilon_0 \text{ and } d(x_{n_k}, x_{m_{k-1}}) \le \varepsilon_0 \text{ for all } m_k > n_k \ge k. \tag{2.1}$$

We consider, for any $m_k > n_k \ge k$,

$$\varepsilon_0 < d(x_{n_k}, x_{m_k}) \le K(d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})) \le K\varepsilon_0 + Kd(x_{m_{k-1}}, x_{m_k}).$$

Taking limit superior form k to infinity, we have

$$\varepsilon_0 \le \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \le K\varepsilon_0. \tag{2.2}$$

Since $d(x_{n_{k-1}}, x_{m_{k-1}}) \le K(d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_{k-1}})) \le Kd(x_{n_{k-1}}, x_{n_k}) + K\varepsilon_0$, taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_{k-1}}, x_{m_{k-1}}) \le K\varepsilon_0. \tag{2.3}$$

If $d(x_{n_{k_0-1}}, x_{m_{k_0-1}}) = 0$ for some $k_0 \in \mathbb{N}$, then

$$\begin{aligned} 0 &< \tilde{n}(2Kd(x_{n_{k_0}}, x_{m_{k_0}}), d(x_{n_{k_0-1}}, x_{m_{k_0-1}}) + d(x_{n_{k_0-1}}, x_{n_{k_0}}) + d(x_{m_{k_0-1}}, x_{m_{k_0}})) \\ &\le [d(x_{n_{k_0-1}}, x_{m_{k_0-1}}) + d(x_{n_{k_0-1}}, x_{n_{k_0}}) + d(x_{m_{k_0-1}}, x_{m_{k_0}})] - 2Kd(x_{n_{k_0}}, x_{m_{k_0}}). \end{aligned}$$

Then

$$\begin{aligned} 2Kd(x_{n_{k_0}}, x_{m_{k_0}}) &\le d(x_{n_{k_0-1}}, x_{m_{k_0-1}}) + d(x_{n_{k_0-1}}, x_{n_{k_0}}) + d(x_{m_{k_0-1}}, x_{m_{k_0}}) \\ &\le d(x_{n_{k_0-1}}, x_{n_{k_0}}) + d(x_{m_{k_0-1}}, x_{m_{k_0}}). \end{aligned}$$

Taking limit superior form k_0 to infinity, we have

$$2K\varepsilon_0 \le \limsup_{k_0\to\infty} 2Kd(x_{n_{k_0}}, x_{m_{k_0}}) \le \limsup_{k_0\to\infty} d(x_{n_{k_0-1}}, x_{n_{k_0}}) + \limsup_{k_0\to\infty} d(x_{m_{k_0-1}}, x_{m_{k_0}}).$$

We get that

$$2K\varepsilon_0 \le \limsup_{k_0\to\infty} 2Kd(x_{n_{k_0}}, x_{m_{k_0}}) \le 0,$$

which contradict to $2K\varepsilon_0 > 0$. Therefore $x_{n_{k-1}} \ne x_{m_{k-1}}$ for all $k \in \mathbb{N}$. By generalized R'-contraction mapping,

$$\begin{aligned} 0 &< \tilde{n}(2Kd(x_{n_k}, x_{m_k}), d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k})) \\ &\le [d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k})] - 2Kd(x_{n_k}, x_{m_k}). \end{aligned}$$

So, we have

$$2Kd(x_{n_k}, x_{m_k}) \le d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k}) \text{ for all } k_0 \in □. \qquad (2.4)$$

By (2.2), (2.3) and (2.4), we get that

$$\begin{aligned} 2K\varepsilon_0 &\le \limsup_{k\to\infty} 2Kd(x_{n_k}, x_{m_k}) \\ &\le \limsup_{k\to\infty} [d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k})] \\ &\le K\varepsilon_0 \\ &\le 2K\varepsilon_0. \end{aligned}$$

Thus $\limsup_{k\to\infty} 2Kd(x_{n_k}, x_{m_k}) = \limsup_{k\to\infty} [d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) + d(x_{m_{k-1}}, x_{m_k})] = 2K\varepsilon_0.$

Since $2K\varepsilon_0 < 2Kd(x_{n_k}, x_{m_k}),$ for all $k_0 \in □$ and the condition $(ñ_2),$ $2K\varepsilon_0 = 0.$ That is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, there exists $z \in X$ such that $x_n \to z.$ By definition of convergence sequence,

for any $\varepsilon > 0$ there exists N such that $d(x_n, z) < \varepsilon$ for all $n > N.$ $\qquad (2.5)$

Next, we will show that z is fixed point. Let $\Omega := \{n \in □ : d(x_n, z) = 0\}.$ Assume that Ω is not finite, then we can find $n_0 > N$ such that $d(x_{n_0}, z) = 0$ i.e. $x_{n_0} = z.$ Sine $x_{n_0} \ne x_{n_0+1}$ and $x_{n_0+1} = Tx_{n_0} = Tz, z \ne Tz.$ Let $\varepsilon = \dfrac{d(z,Tz)}{2} > 0.$ By $(2.5),$ we have

$$\varepsilon > d(x_{n_0+1}, z) = d(Tx_{n_0}, z) = d(Tz, z) = 2\varepsilon,$$

which is a contradiction. Therefore Ω is finite, there exists n_0 such that $d(x_n, z) > 0$ for all $n > n_0.$ Since T is a generalized R'-contraction mapping,

$$\begin{aligned} 0 &< ñ(2Kd(Tx_n, Tz), d(x_n, z) + d(x_n, Tx_n) + d(z, Tz)) \\ &\le [d(x_n, z) + d(x_n, Tx_n) + d(z, Tz)] - 2Kd(Tx_n, Tz). \end{aligned}$$

Hence,

$$\begin{aligned} 2Kd(Tx_n, Tz) &\le d(x_n, z) + d(x_n, Tx_n) + d(z, Tz) \\ &\le d(x_n, z) + d(x_n, Tx_n) + K[d(z, Tx_n) + d(Tx_n, Tz)] \\ &= d(x_n, z) + d(x_n, x_{n+1}) + Kd(z, x_{n+1}) + Kd(Tx_n, Tz) \\ &\le d(x_n, z) + d(x_n, x_{n+1}) + Kd(z, x_{n+1}). \end{aligned}$$

Taking limit n to infinity,

$$\lim_{n\to\infty} Kd(Tx_n, Tz) \le \lim_{n\to\infty} d(x_n, z) + \lim_{n\to\infty} d(x_n, x_{n+1}) + \lim_{n\to\infty} Kd(z, x_{n+1}) = 0.$$

Thus $\lim_{n\to\infty} Kd(Tx_n, Tz) = 0.$

That is $\{x_{n+1} = Tx_n\} \to Tz.$ By the uniqueness of the limit, $Tz = z.$ Finally, let us show that z is unique fixed point of $T.$ Assume $x = Tx$ and $y = Ty$ such that $x \ne y.$ Let $a_n = d(x,y) > 0$ for all $n \in □.$ By assumption, we have

$$\begin{aligned} ñ(2Ka_{n+1}, 2a_n + a_{n+1}) &= ñ(2Kd(x,y), d(x,y) + d(x,Tx) + d(y,Ty)) \\ &> 0. \end{aligned}$$

By using $(ñ_1),$ we get $a_n \to 0,$ which imply that $d(x,y) = 0,$ which is a contradiction. So $x = y.$ □

Theorem 2.2. Let (X,d) be a complete b-metric space with coefficient $K \geq 1,$ and suppose that let $T : X \to X$ be generalized R'-Max-kannan mapping , i.e.,

$$\tilde{n}(2Kd(Tx,Ty), \max\{d(x,y), d(x,Tx), d(y,Ty)\}) > 0$$

with respect $\tilde{n} \in R^*$. If $\tilde{n}(2Kt,s) \leq s - 2Kt$ for all $s,t \in (0,\infty)$ then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a arbitrary point. Let $\{x_n\}$ be Picard sequence of T based on x_0, that is, $x_{n+1} = Tx_n$ for all $n \geq 1.$ If there exists $n_0 \in \square$ such that $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ which

Implies that x_{n_0} is a fixed point. Assume $x_n \neq x_{n+1}$ for all $n \in \square$. Let $\{a_n\} \subset (0,\infty)$ be a sequence defined by $a_n = d(x_n, x_{n+1}) > 0$ for all $n \in \square$. By R'-Max-kannan mapping,

$$\begin{aligned}\tilde{n}(2Ka_{n+1}, a_n) &= \tilde{n}(2Kd(x_{n+1}, x_{n+2}), \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\ &= \tilde{n}(2Kd(Tx_n, Tx_{n+1}), \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\}) \\ &> 0.\end{aligned}$$

Case I If $\max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} = d(x_n, x_{n+1})$ then

$$\begin{aligned}0 &< \tilde{n}(2Ka_{n+1}, a_n) \\ &= \tilde{n}(2Kd(x_{n+1}, x_{n+2}), d(x_n, x_{n+1}))\end{aligned}$$

Case II If $\max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} = d(x_{n+1}, Tx_{n+1})$ then

$$\begin{aligned}0 &< \tilde{n}(2Kd(x_{n+1}, x_{n+2}), d(x_{n+1}, Tx_{n+1})) \\ &< \tilde{n}(2Kd(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) \\ &< d(x_{n+1}, x_{n+2}) - 2Kd(x_{n+1}, x_{n+2})) \\ &= a_{n+1} - 2Ka_{n+1} \\ &< 0,\end{aligned}$$

which is a contradiction. Thus

$$\begin{aligned}\tilde{n}(2Ka_{n+1}, a_n) &= \tilde{n}(2Kd(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) \\ &> 0.\end{aligned}$$

By using the condition $(\tilde{n}_1'),$ we get that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} a_n = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence reasoning by contradiction. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon_0 > 0$ such that

$$d(x_{n_k}, x_{m_k}) > \varepsilon_0 \text{ and } d(x_{n_k}, x_{m_{k-1}}) \leq \varepsilon_0 \text{ for all } m_k > n_k \geq k. \tag{2.6}$$

We consider, for any $m_k > n_k \geq k,$

$$\varepsilon_0 < d(x_{n_k}, x_{m_k}) \leq K(d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})) < K(\varepsilon_0 + d(x_{m_{k-1}}, x_{m_k})).$$

Taking limit superior from k to infinity, we have

$$\varepsilon_0 \leq \limsup_{k\to\infty} d(x_{n_k}, x_{m_k}) \leq K\varepsilon_0. \tag{2.7}$$

Since $d(x_{n_{k-1}}, x_{m_{k-1}}) \leq K(d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_{k-1}})),$ taking limit superior from k to infinity,

$$\limsup_{k\to\infty} d(x_{n_{k-1}}, x_{m_{k-1}}) \leq K\varepsilon_0. \tag{2.8}$$

If $d(x_{n_{k_0-1}}, x_{m_{k_0-1}}) = 0$ for some $k_0 \in \square$, then

$$0<\tilde{n}(2Kd(x_{n_{k_0}},x_{m_{k_0}}),\max\{d(x_{n_{k_0-1}},x_{m_{k_0-1}}),d(x_{n_{k_0-1}},x_{n_{k_0}}),d(x_{m_{k_0-1}},x_{m_{k_0}})\})$$
$$\leq \max\{d(x_{n_{k_0-1}},x_{m_{k_0-1}}),d(x_{n_{k_0-1}},x_{n_{k_0}}),d(x_{m_{k_0-1}},x_{m_{k_0}})\}-2Kd(x_{n_{k_0}},x_{m_{k_0}}).$$

Then $2Kd(x_{n_{k_0}},x_{m_{k_0}})\leq \max\{d(x_{n_{k_0-1}},x_{m_{k_0-1}}),d(x_{n_{k_0-1}},x_{n_{k_0}}),d(x_{m_{k_0-1}},x_{m_{k_0}})\}$
$$\leq \max\{d(x_{n_{k_0-1}},x_{n_{k_0}}),d(x_{m_{k_0-1}},x_{m_{k_0}})\}.$$

Taking limit superior form k_0 to infinity, we have

$$2K\varepsilon_0 \leq \limsup_{k_0\to\infty} 2Kd(x_{n_{k_0}},x_{m_{k_0}})$$
$$\leq \limsup_{k_0\to\infty}\max\{d(x_{n_{k_0-1}},x_{n_{k_0}}),d(x_{m_{k_0-1}},x_{m_{k_0}})\}n$$
$$\leq \max\{\limsup_{k_0\to\infty} d(x_{n_{k_0-1}},x_{n_{k_0}}),\limsup_{k_0\to\infty} d(x_{m_{k_0-1}},x_{m_{k_0}})\}$$
$$\leq 0,$$

which contradict to $2K\varepsilon_0\leq 0$. Therefore $x_{n_{k-1}}\neq x_{m_{k-1}}$ for all $k\in\mathbb{N}$. By generalized R'-Max-kannan mapping,

$$0<\tilde{n}(2Kd(x_{n_k},x_{m_k}),\max\{d(x_{n_{k-1}},x_{m_{k-1}}),d(x_{n_{k-1}},x_{n_k}),d(x_{m_{k-1}},x_{m_k})\})$$
$$\leq \max\{d(x_{n_{k-1}},x_{m_{k-1}}),d(x_{n_{k-1}},x_{n_k}),d(x_{m_{k-1}},x_{m_k})\}-2Kd(x_{n_k},x_{m_k}).$$

It implies that

$$2Kd(x_{n_k},x_{m_k})\leq \max\{d(x_{n_{k-1}},x_{m_{k-1}}),d(x_{n_{k-1}},x_{n_k}),d(x_{m_{k-1}},x_{m_k})\} \text{ for all } k_0\in\mathbb{N}. \quad (2.9)$$

By (2.7), (2.8) and (2.9), we get that

$$2K\varepsilon_0\leq \limsup_{k\to\infty} 2Kd(x_{n_k},x_{m_k})\leq \limsup_{k\to\infty}[\max\{d(x_{n_{k-1}},x_{m_{k-1}}),d(x_{n_{k-1}},x_{n_k}),d(x_{m_{k-1}},x_{m_k})\}].$$

Since $0\leq K\varepsilon_0\leq 0$, we have $K\varepsilon_0=0$. That is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, there exists $z\in X$ such that $x_n\to z$. By definition of convergence sequence,

for any $\varepsilon>0$ there exists N such that $d(x_n,z)<\varepsilon$ for all $n > N$. $\quad (2.10)$

Next, we will show that z is fixed point. Let $\Omega := \{n\in\mathbb{N} : d(x_n,z)=0\}$. Assume that Ω is not finite, then we can find $n_0>N$ such that $d(x_{n_0},z)=0$ i.e. $x_{n_0}=z$. Sine $x_{n_0}\neq x_{n_0+1}$ and $x_{n_0+1}=Tx_{n_0}=Tz, z\neq Tz$. Let $\varepsilon=\dfrac{d(z,Tz)}{2}>0$. By (2.10), we have

$$\varepsilon>d(x_{n_0+1},z)=d(Tx_{n_0},z)=d(Tz,z)=2\varepsilon,$$

which is a contradiction. Therefore Ω is finite, there exists n_0 such that $d(x_n,z)>0$ for all $n>n_0$. Since T is a generalized R'-Max-kannan mapping,

$$0<\tilde{n}(2Kd(Tx_n,Tz),\max\{d(x_n,z),d(x_n,Tx_n),d(z,Tz)\})$$
$$\leq \max\{d(x_n,z),d(x_n,Tx_n),d(z,Tz)\}-2Kd(Tx_n,Tz).$$

Hence, $2Kd(Tx_n,Tz)\leq \max\{d(x_n,z),d(x_n,x_{n+1}),d(z,Tz)\}$.

If $\max\{d(x_n,z),d(x_n,x_{n+1}),d(z,Tz)\}=d(z,Tz)$, then

$$2Kd(Tx_n,Tz) \leq Kd(z,Tx_n)+Kd(Tx_n,Tz).$$

So,

$$Kd(Tx_n,Tz)\leq Kd(z,Tx_n).$$

Taking limit n to infinity,

$$0 \le \lim_{n\to\infty} Kd(Tx_n, Tz) \le \lim_{n\to\infty} Kd(z, Tx_n) = 0.$$

If $\max\{d(x_n,z), d(x_n,x_{n+1}), d(z,Tz)\} = d(x_n,z),$ then

$$2Kd(Tx_n,Tz) \le d(x_n,z).$$

Taking limit n to infinity, thus

$$\lim_{n\to\infty} 2Kd(Tx_n,Tz) \le \lim_{n\to\infty} d(x_n,z) = 0.$$

That is $\{x_{n+1} = Tx_n\} \to Tz.$ By the uniqueness of the limit, $Tz = z.$ Finally, we show that z is unique fixed point of $T.$ Assume $x = Tx$ and $y = Ty$ such that $x \ne y.$ Let $a_n = d(x,y) > 0$ for all $n \in \Box$. We consider

$$0 < \tilde{n}(2Kd(Tx,Ty), \max\{d(x,y), d(x,Tx), d(y,Ty)\})$$
$$< d(x,y) - 2kd(x,y)$$
$$\Rightarrow \quad 2kd(x,y) < d(x,y)$$
$$\Rightarrow \quad (2k-1)d(x,y) < 0$$
$$\Rightarrow \quad 0 \le d(x,y) < 0$$

So $x = y.$ □

Conclusions

The purpose of this paper is to extend and improve some results concerning of R'-Max-kannan and R''-kannan mappings to generalize R'-contraction and generalized R'-Kannan mappings. Second, we establish new mapping, that is generalized R'-contraction and generalized R'-Kannan mappings and prove the results of fixed point for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces. Moreover, we obtain fixed point theorems for generalized R'-contraction and generalized R'-Kannan mappings in b-metric spaces as follows:

1. Let K be a given real number such that $K \ge 1.$ A function $\tilde{n} : [0,\infty) \times [0,\infty) \to \Box$ is called **generalized** R**-function** if it satisfies the following two conditions:

$(\tilde{n}_1)$ If $\{a_n\} \subset (0,\infty)$ is a sequence such that $\tilde{n}(2Ka_{n+1}, 2a_n + a_{n+1}) > 0$ for all $n \in \Box$, then $a_n \to 0.$

$(\tilde{n}_2)$ If $\{a_n\},\{b_n\} \subset (0,\infty)$ are two sequences such that $\limsup_{n\to\infty} Ka_n = \limsup_{n\to\infty} b_n = L \ge 0$ and verifying that $L < Ka_n$ and $\tilde{n}(Ka_n, b_n) > 0$ for all $n \in \Box$, then $L = 0.$

2. Let X be a non-empty set. A b-metric on a set X is a mapping $d : X \times X \to [0,+\infty)$ satisfying the following conditions: for any $x,y,z \in X$,

(b_1) $d(x,y) = 0$ if and only if $x = y$,

(b_2) $d(x,y) = d(y,x);$

(b_3) there exists $K \ge 1$ such that $d(x,y) \le K(d(x,z) + d(z,y)).$

Then (X,d) is known as a b-metric spaces with coefficient $K.$

3. Let (X,d) be a complete b-metric space with coefficient $K \geq 1,$ and suppose that let $T: X \to X$ be generalized R'-contraction mapping, i.e.

$$\tilde{n}(2Kd(Tx,Ty), d(x,y)+d(x,Tx)+d(y,Ty)) > 0$$

with respect $\tilde{n} \in R^*$. If $\tilde{n}(2Kt,s) \leq s - 2Kt$ for all $s,t \in (0,\infty)$ then T has a unique fixed point.

4. Let (X,d) be a complete b-metric space with coefficient $K \geq 1,$ and suppose that let $T: X \to X$ be generalized R'-Max-kanan mapping , i.e.,

$$\tilde{n}(2Kd(Tx,Ty), \max\{d(x,y), d(x,Tx), d(y,Ty)\}) > 0$$

with respect $\tilde{n} \in R^*$. If $\tilde{n}(2Kt,s) \leq s - 2Kt$ for all $s,t \in (0,\infty)$ then T has a unique fixed point.

5. Discussion

Future research directions may also be possible.

Open problems 1:

If T satisfies

$$\tilde{n}(5Kd(Tx,Ty), \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}) > 0$$

then T has a unique fixed point.

Open problems 2:

If T satisfies

$$\tilde{n}(5Kd(Tx,Ty), \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}) > 0$$

then T has a unique fixed point.

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