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CONVERGENCE THEOREMS FOR A BIVARIATE NONEXPANSIVE OPERATOR

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Abstract. In this paper, we prove some fixed point theorems for coupled-nonexpansive mapping and prove strong convergence and weakly convergence theorems for a double Mann-type iterative process to approximating a fixed point for coupled-nonexpansive operator in Hilbert spaces. Moreover, we prove some properties of the coupled fixed point set for coupled-nonexpansive mapping and prove fixed point theorem for such mapping on Banach spaces.

Keywords: fixed point; coupled-nonexpansive; strong convergence; weakly convergence; mann iterative.

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1. Introduction

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Let $(X,||\cdot||)$ be a real Banach space and let K be a nonempty subset of X. A mapping $T:K\to K$ is said to be nonexpansive, if $||Tx-Ty||\leq||x-y||$, for each $x,y\in K$. (see [1]). During last four decades many authors have investigated nonexpansive mappings and the set of its fixed points. We now review the needed definitions and results. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{R}$ the set of all real numbers, respectively. A nonempty subset $K\subseteq X$ is said to be *convex*, if $\alpha x+(1-\alpha)y\in K$ for all $x,y\in K$ and $\alpha\in[0,1]$. A Banach space K be *strictly convex* if $\|\frac{x+y}{2}\|<1$ for each $x,y\in K$ with $||x||=||y||=1$ and $x\neq y$. A Banach space K be *uniformly convex* if for any $\varepsilon\in(0,2]$ there exists $\delta=\delta(\varepsilon)>0$, whenever $x,y\in K$, $||x||\leq 1, ||y||\leq 1$ and $||x-y||\geq\varepsilon$ then $||\frac{x+y}{2}||<1-\delta$. It is clear that uniform convexity implies strict convexity (see [2]). A mapping $T:K\to K$ have a coupled fixed point, if there exist $x,y\in X$ such that $T(x,y)=x$ and $T(y,x)=y$. Let $\{x_n\}$ be a bounded sequence in a Banach space $(X,||\cdot||)$. For $x\in X$, we define a continuous functional $r(\cdot,x_n):X\to[0,\infty)$ by $r(x,x_n)=\limsup_{n\to\infty}||x-x_n||$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\})=\inf\{r(x,x_n):x\in X\}$. The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K\subseteq X$ is the set $A_K(\{x_n\})=\{x\in X: r(x,x_n)\leq r(y,x_n),\ \forall y\in K\}$. This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot,x_n)$ in K. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$ (see [3]).

Lemma 1. [3] *Let $(X,||\cdot||)$ be a uniformly convex Banach space with modulus of convexity of δ. Then every bounded sequence $\{x_n\}$ in K has a unique asymptotic center in K.*

The Banach fixed point theorem concerns certain mappings of a complete metric space itself. It states conditions sufficient for the existence and uniqueness of a fixed point and it's also given a constructive procedure for obtaining better and better approximations to the fixed point, this is a method such that we choose x_0 in a given set and calculate recursively a sequence $x_0,x_1,x_2,...$ from a relation of the form $x_n=Tx_{n-1}=T^nx_0,\quad\forall n\geq 1$. It is also know as the *Picard iteration* starting at x_o. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings have been studied many mathematicians as follows:

Definition 2. Let $(X, ||\cdot||)$ be a normed space and $K \subseteq X$ be a closed and convex. Three classical iteration processes are often used to approximate a fixed point of a nonlinear mapping $T : K \to K$.

Krasnoselskij's iteration

The first one is introduced by Schaefer [4] which is defined as follows: $x_0 \in K$

$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n, \;\; n \geq 0,$$

where $\lambda \in (0,1)$.

Halpern's iteration

The second one is introduced by Halpern [5] which is defined as follows: $x_0 \in K$

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \;\; n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0,1]$.

Mann's iteration

The third one is introduced by Mann [6] which is defined as follows: $x_0 \in K$

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \;\; n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0,1]$.

Next, Takahashi [2] proved fixed point theorems for nonexpansive mappings in Hilbert spaces as follows:

Theorem 3. *Let K be a nonempty closed and convex subset of a Hilbert space H and let $F : K \to K$ be a nonexpansive. Then the following are equivalent:*
(i) The set $Fix(F)$ of fixed points of T is nonempty;
(ii) $\{F^n x\}$ is bounded for some $x \in K$.

In 2006, Bhaskar and Lakshmikantham [7] established a fixed point theorem for mixed monotone mappings in partially ordered metric spaces. Moreover, the double sequence $\{(x_n, y_n)\}n \geq 0$, defined by the Picard-type iteration $x_{n+1} = T(x_n, y_n), \quad y_{n+1} = T(y_n, x_n), \quad n \geq 0$, with $x_0, y_0 \in X$, is convergent and its limit is always a coupled fixed point of F. In 2013, Olaoluwa et al.[8] introduced the definitions of nonexpansive condituon for coupled maps in product s-paces and proved the existence of coupled fixed points of such mappings when X is a uniformly convex as follows:

Definition 4. Let X be a Banach spaces and K be a nonempty subset of X. A mapping $T : K \times K \to K$ is said to be *coupled-nonexpansive* if

$$||T(x,y) - T(u,v))|| \leq \frac{1}{2}(||x-u|| + ||y-v||), \tag{1}$$

for all $x, y, u, v \in K$.

Throughout this paper, a mapping $T : K \times K \to K$ is call *bivariate nonexpansive* or *coupled-nonexpansive* if T satisfies 1.

Example 5. Let $X = \mathbb{R}$. Defined

$$||x|| = |x|,$$

for every $x \in \mathbb{R}$ and $T : X \times X \to X$ be defined by $T(x,y) = \frac{x-y}{2}$, for all $x, y \in X$. Indeed for all $x, y \in X$, we consider

$$\begin{aligned} ||T(x,y) - T(u,v)|| &= |\frac{x-y}{2} - \frac{u-v}{2}| \\ &= \frac{1}{2}|(x-y) - (u-v)| \\ &\leq \frac{1}{2}(|x-y| + |u-v|) \\ &= \frac{||x-u|| + ||y-v||}{2}. \end{aligned}$$

Hence T is coupled-nonexpansive mapping.

Next, Berinde et al.[9] proved weak and strong convergence theorems for a double Krasnoselskij-type iterative method to approximate coupled solutions of a bivariate nonexpansive operator $T : K \times K \to K$, where K is a nonempty closed and convex subset of a Hilbert space as follows:

Definition 6. A mapping $T : K \times K \to K$ is called *demicompact* if it has the property that whenever $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K with the property that $\{T(u_n, v_n) - u_n\}$ and $\{T(v_n, u_n, \ ) - v_n\}$ converge strongly to 0, then there exists a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n, v_n)\}$ such that $u_{n_k} \to u$ and $v_{n_k} \to v$ strongly.

Theorem 7. *Let K be a bounded, closed and convex subset of a Hilbert space H and let $T : K \times K \to K$ be weakly nonexpansive and demicompact operator. Then the set of coupled fixed points of T is nonempty and the double iterative algorithm $\{(x_n, x_n)\}_{n=0}^{\infty}$ given by x_0 in K and*

$$x_{n+1} = \lambda x_n + (1-\lambda)T(x_n, x_n), \ \ n \geq 0,$$

where $\{\lambda\}_{n=0}^{\infty} \in (0,1)$, converges (strongly) to a coupled fixed point of T.

In this paper, we prove some fixed point theorems for coupled-nonexpansive mapping and prove strong convergence and weakly convergence theorems for a double Mann-type iterative process to approximating a fixed point for coupled-nonexpansive operator in Hilbert spaces. Moreover, we prove some properties of the coupled fixed point set for coupled-nonexpansive mapping and prove fixed point theorem for such mapping on Banach spaces.

2. Fixed Point Theorems

In this section, we prove fixed point theorems for coupled-nonexpansive mapping and prove strong convergence theorems in Banach spaces.

Theorem 8. *Let K be a nonempty closed and convex subset of a Hilbert space H and let $T : K \times K \to K$ be a coupled-nonexpansive. Then the following are equivalent:*
(i) The set $F_c(T)$ of fixed points of T is nonempty;
(ii) $\{T^n(x,x)\}$ is bounded for some $(x,x) \in K \times K$.

Proof. Let $F : K \to K$ be given by $Fx = T(x,x)$, for all $x \in K$. By the coupled-nonexpansiveness property of T, we obtain the nonexpansiveness of F and hence, by Theorem 3, it follows that $F_c(T) \neq \emptyset$. □

3. The Properties of Coupled-Fixed Point Set

In this section, we will prove some properties of coupled-fixed point set for coupled-nonexpansive mapping in a Banach space. Let $(X, ||\cdot||)$ be a Banach space. and let K be a nonempty subset of X. We will denote *the coupled fixed point set* of a mapping T by $F_c(T) = \{(x,y) \in K \times K : T(x,y) = x \text{ and } T(y,x) = y\}$.

Lemma 9. *Let K be a nonempty bounded closed convex subset of strictly Banach spaces X, with $||(x,y)||_{X^2} = ||x|| + ||y||$ for all $x, y \in X$. Let $T : K \times K \to K$ be coupled-nonexpansive and $F_c(T) \neq \emptyset$, then $F_c(T)$ are closed and convex.*

Proof. Suppose that $\{x_n\}$ is a sequence in $F_c(T)$ which converges to some $x \in K$, where $x_n = (y_n, z_n)$ and $x = (y,z)$. Then $||y_n - y|| + ||z_n - z|| = ||x_n - x||_{X^2} \to 0$ as $n \to \infty$. Then $y_n \to y$ and $z_n \to z$. We will to show that $x \in F_c(T)$. We consider

$$||y_n - Tx|| = ||Tx_n - Tx|| = ||T(y_n, z_n) - T(y,z)|| \leq \frac{1}{2}||y_n - y|| + \frac{1}{2}||z_n - z|| \tag{2}$$

and

$$||z_n - T(z,y)|| = ||T(z_n, y_n) - T(z,y)|| \leq \frac{1}{2}||z_n - z|| + \frac{1}{2}||y_n - y||. \tag{3}$$

So

$$\lim_{n\to\infty} ||y_n - Tx|| \leq \lim_{n\to\infty} \frac{1}{2}||y_n - y|| + \lim_{n\to\infty} \frac{1}{2}||z_n - z|| = 0$$

and

$$\lim_{n\to\infty} ||z_n - T(z,y)|| \leq \lim_{n\to\infty} \frac{1}{2}||z_n - z|| + \lim_{n\to\infty} \frac{1}{2}||y_n - y|| = 0.$$

Thus $T(y,z) = y$ and $T(z,y) = z$ by the uniqueness of limit point. Hence $F(T)$ is closed. Next, we will to show that $F_c(T)$ is convex, let $u, v \in F_c(T)$ and each $\alpha \in [0,1]$, where $u = (u_1, u_2)$, $v = (v_1, v_2)$. Now, to show that $w = \alpha u + (1-\alpha)v \in F(T)$. Let $w = (w_1, w_2)$. Then $w_1 = \alpha u_1 + (1-\alpha)v_1$ and $w_2 = \alpha u_2 + (1-\alpha)v_2$. Since

$$||u_1 - v_1|| = ||Tu - Tv|| \leq \frac{1}{2}(||u_1 - y_1|| + ||u_2 - v_2||)$$

and

$$||u_2 - v_2|| = ||T(u_2, v_1) - T(y_2, v_1)|| \leq \frac{1}{2}(||u_2 - v_2|| + ||u_1 - v_1||),$$

we have $||u_1 - v_1|| \leq ||u_2 - v_2||$ and $||u_1 - v_1|| \geq ||u_2 - v_2||$. Thus,

$$||u_1 - v_1|| = ||u_2 - v_2||. \tag{4}$$

Since

$$||u_1 - w_1|| = ||u_1 - (\alpha u_1 + (1-\alpha)v_1)|| = (1-\alpha)||u_1 - v_1||$$

and

$$||u_2 - w_2|| = ||u_2 - (\alpha u_2 + (1-\alpha)v_2)|| = (1-\alpha)||u_2 - v_2||,$$

we get

$$||u_1 - w_1|| = ||u_2 - w_2||. \tag{5}$$

Since

$$||v_1 - w_1|| = ||v_1 - (\alpha u_1 + (1-\alpha)v_1)|| = \alpha||u_1 - v_1||$$

and

$$||v_2 - w_2|| = ||v_2 - (\alpha u_2 + (1-\alpha)v_2)|| = \alpha||u_2 - v_2||,$$

it follows that

$$||v_1 - w_1|| = ||v_2 - w_2||. \tag{6}$$

Similar to the above proof, it follows that

$$||u_1 - Tw|| = ||Tu - Tw|| \leq \frac{1}{2}(||u_1 - w_1|| + ||u_2 - w_2||) = ||u_1 - w_1||,$$

$$||u_2 - T(w_2, w_1)|| \leq \frac{1}{2}(||u_2 - w_2|| + ||u_1 - w_1||) = ||u_2 - w_2||,$$

$$||v_1 - Tw|| = ||Tv - T(z_1, z_2)|| \leq \frac{1}{2}(||v_1 - w_1|| + ||v_2 - w_2||) = ||v_1 - w_1||,$$

and

$$||v_2 - T(w_2, w_1)|| = ||T(v_2, v_1) - T(w_2, w_1)|| \leq \frac{1}{2}(||v_2 - w_2|| + ||v_1 - w_1||) = ||v_2 - w_2||.$$

For $w = \alpha u + (1-\alpha)v$ where $w = (w_1, w_2)$, we consider

$$\begin{aligned} ||u_1 - v_1|| &\leq ||u_1 - Tw|| + ||Tw - v_1|| \leq ||u_1 - w_1|| + ||w_1 - v_1|| \\ &= ||u_1 - (\alpha u_1 + (1-\alpha)v_1)|| + ||(\alpha u_1 + (1-\alpha)v_1) - v_1|| \\ &= ||u_1 - v_1||, \end{aligned} \tag{7}$$

and

$$\begin{aligned} ||u_2 - v_2|| &\le ||u_2 - T(w_2,w_1)|| + ||T(w_2,w_1) - v_2|| \\ &\le ||u_2 - w_2|| + ||w_2 - v_2|| \\ &= ||u_2 - (\alpha u_2 + (1-\alpha)v_2|| + ||(\alpha u_2 + (1-\alpha)v_2) - v_2|| = ||u_2 - v_2||. \end{aligned} \tag{8}$$

Thus $||u_1 - Tw|| = ||u_1 - w_1||$ and $||Tw - v_1|| = ||w_1 - v_1||$, because if $||u_1 - Tw|| < ||u_1 - w_1||$ or $||Tw - v_1|| < ||w_1 - v_1||$, then which the contradiction to $||u_1 - v_1|| < ||u_1 - v_1||$. Since X is strictly convex, we have $Tw = \alpha u_1 + (1-\alpha)v_1 = w_1$. Similarly, $T(w_2,w_1) = \alpha w_2 + (1-\alpha)v_2 = w_2$, and then w is a coupled fixed point of T, that is $\alpha u + (1-\alpha)v \in F_c(T)$. Hence $F_c(T)$ is convex.

□

Theorem 10. *Let K be a nonempty bounded closed convex subset of uniformly convex Banach space X such that $||(x,y)||_{X^2} = ||x|| + ||y||$ with uniformly convex Banach space with modulus of convexity of δ. Suppose that a map $T : K \times K \to K$ be coupled-nonexpansive and $\{x_n\}$ and $\{y_n\}$ are a sequences in K defined by $x_{n+1} = T(x_n,y_n)$, $y_{n+1} = T(y_n,x_n)$ and $||x_n - z||$, $||y_n - z||$ are increase sequences in $\mathbb{R}$ for all $z \in K$ with $||x_{n-1} - x|| = ||y_{n-1} - y||$ for all $n \in \mathbb{N}$. Then $F_c(T)$ are nonempty, closed and convex.*

Proof. By Lemma 1, the asymptotic center of any bounded sequence in K, particularly, the asymptotic center of approximate coupled fixed point sequence for T is in K. Let $A(\{x_n\}) = \{x\}$ and $A(\{y_n\}) = \{y\}$. We consider

$$\begin{aligned} ||x_n - T(x,y)|| &\le ||x_{n+1} - T(x,y)|| = ||T(x_n,y_n) - T(x,y)|| \\ &\le \frac{1}{2}(||x_n - x|| + ||y_n - y||) = ||x_n - x||, \end{aligned} \tag{9}$$

thus $\limsup_{n\to\infty} ||x_n - T(x,y)|| \le \limsup_{n\to\infty} ||x_n - x||$. Similarly,

$$\begin{aligned} ||y_n - T(y,x)|| &\le ||y_{n+1} - T(y,x)|| = ||T(y_n,x_n) - T(y,x)|| \\ &\le \frac{1}{2}(||y_n - x|| + ||x_n - y||) = ||y_n - y||, \end{aligned} \tag{10}$$

hence $\limsup_{n\to\infty}||y_n - T(y,x)|| \leq \limsup_{n\to\infty}||y_n - y||$. By the uniqueness of the asymptotic center, $T(x,y) = x$ and $T(y,x) = y$. Hence $F_c(T)$ is nonempty. By Lemmas 9,we conclude that $F_c(T)$ are nonempty, closed and convex. □

4. Iterative Approximation of Fixed Points

Theorem 11. *Let K be a nonempty, bounded, closed and convex subset of a Hilbert space H and let $T : K \times K \to K$ be coupled-nonexpansive operator. Then the Mann iterative $\{x_n\}_{n=0}^{\infty}$ given by x_0 in K and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n, x_n), \;\; n \geq 0,$$

(11)

where $\{\alpha_n\}_{n=0}^{\infty} \subseteq (0,1)$, weakly converges to coupled fixed point of T.

Proof. As in proof of Theorem 8, for each $w \in Fix(T)$ and each n, we have,

$$||x_{n+1} - w|| \leq ||x_n - w||.$$

Defined $g : Fix(T) \to [0,\infty)$ by $g(w) = \lim_{n\to\infty}||x_n - w||$. we see that g is well defined and is a lower semi-continuous convex function on the nonempty convex set $Fix(F)$. Let $r = \inf\{g(w) : w \in Fix(F)\}$. For each $\varepsilon > 0$, the set $M_\varepsilon = \{z : g(z) \leq r + \varepsilon\}$ is closed and convex and then weakly compact. Therefore $\cap_{\varepsilon>0} M_\varepsilon$ in fact $\varepsilon > 0$, $M_\varepsilon = z : g(z) = r \equiv L$. Moreover, L contains exactly one point. Indeed, since L is convex and closed, for $w_1, w_2 \in L$, and $w_{\alpha_n} = \alpha_n w_1 + (1-$

$\alpha_n)w_2$, we get

$$\begin{aligned}
g^2(w_{\alpha_n}) &= \lim_{n\to\infty} ||w_{\alpha_n} - x_n||^2 \\
&= \lim_{n\to\infty} ||\alpha_n w_1 + (1-\alpha_n)w_2 - x_n||^2 \\
&= \lim_{n\to\infty} (\alpha_n^2 ||w_1 - x_n||^2 + (1-\alpha_n)^2 ||w_2 - x_n||^2 \\
&\quad + \alpha_n(1-\alpha_n)\langle w_1 - x_n, w_2 - x_n\rangle) \\
&\leq \lim_{n\to\infty} (\alpha_n^2 ||w_1 - x_n||^2 + (1-\alpha_n)^2 ||w_2 - x_n||^2 \\
&\quad + \alpha_n(1-\alpha_n)||w_1 - x_n||\ ||w_2 - x_n||) \\
&\quad + \lim_{n\to\infty} \alpha_n(1-\alpha_n)[\langle w_1 - x_n, w_2 - x_n\rangle - ||w_1 - x_n||\ ||w_2 - x_n||]. \qquad (12)
\end{aligned}$$

So $\lim_{n\to\infty} \alpha_n(1-\alpha_n)[\langle w_1 - x_n, w_2 - x_n\rangle - ||w_1 - x_n||\ ||w_2 - x_n||] = 0$. Since

$$\lim_{n\to\infty} ||w_1 - x_n|| = r = r \lim_{n\to\infty} ||w_2 - x_n||,$$

we have

$$\begin{aligned}
||w_1 - w_2||^2 &= ||w_1 - x_n - (w_2 - x_n)||^2 \\
&= ||w_1 - x_n||^2 + ||w_2 - x_n||^2 - 2\langle w_1 - x_n, w_2 - x_n\rangle \to r^2 + r^2 - 2r^2 = 0, \qquad (13)
\end{aligned}$$

a contradiction. Now, we will show that $x_n = F^n(x_0, x_0) \rightharpoonup w_1$, it suffices to assume that $x_{n_j} \rightharpoonup w$ for an infinite subsequence and then prove that $w = w_1$. By the arguments in the proof of Theorem 8, $w \in Fix(F)$. Considering the definition of g and the fact that $x_{n_j} \to p$, we obtain that

$$\begin{aligned}
||x_{n_j} - w_1||^2 &= ||x_{n_j} - w - (w_1 - w)||^2 \\
&= ||x_{n_j} - w||^2 + ||w_1 - w||^2 - 2\langle x_{n_j} - p, w_1 - p\rangle \to g^2(w) + ||w_1 - w||^2 \\
&= g^2(w_1) = r^2. \qquad (14)
\end{aligned}$$

Since $g^2(w_{\alpha_n}) \geq r^2$, we conclude that $||w - w_1|| \leq 0$. Therefore $w = w_1$. □

Theorem 12. *Let K be a nonempty, bounded, closed, and convex subset of a Hilbert space H and let $T : K \times K \to K$ be coupled-nonexpansive and demicompact operator. Then the set of* *coupled fixed points of T is nonempty and the double iterative algorithm $\{(x_n,x_n)\}_{n=0}^{\infty}$ given by x_0 in K and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n,x_n), \;\; n \geq 0, \tag{15}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0,1]$, strongly converges to a coupled fixed point of T.

Proof. By Theorem 8, T has at least one coupled fixed point with equal components, say $(x',x') \in C \times C$. We will to show that the sequence $\{x_n - T(x_n,x_n)\}$ converges strongly to 0. We consider

$$\begin{aligned} ||x_{n+1}-x'||^2 &= ||\alpha_n x_n + (1-\alpha_n)T(x_n,x_n) - x'||^2 \\ &= \alpha_n^2||x_n - x'||^2 + (1-\alpha_n)^2||T(x_n,x_n)-x'||^2 \\ &\quad + \alpha_n(1-\alpha_n)\langle x_n - x', T(x_n,x_n) - x'\rangle, \end{aligned} \tag{16}$$

and

$$\begin{aligned} ||x_n - T(x_n,x_n)||^2 &= ||x_n - x'||^2 + ||T(x_n,x_n)-x'||^2 \\ &\quad + \langle x_n - x', T(x_n,x_n) - x'\rangle. \end{aligned} \tag{17}$$

Since T coupled-nonexpansive and $T(x',x') = x'$, we obtain

$$||T(x_n,x_n) - x'|| = ||T(x_n,x_n) - T(x',x')|| \leq ||x_n - x'||. \tag{18}$$

Now, by (16),(17) and (18), it follows that for any $\{\beta_n\}$ we get

$$\begin{aligned} ||x_{n+1}-x'||^2 + \beta_n^2||x_n - T(x_n,x_n)||^2 &\leq (2\beta_n^2 + \alpha_n^2 + (1-\alpha_n)^2)||x_n - x'||^2 \\ &\quad + 2(\alpha_n(1-\alpha_n) - \beta_n)\langle T(x_n,x_n) - x', x_n - x'\rangle. \end{aligned} \tag{19}$$

If we choose now a sequence such that $0 \leq \beta_n^2 \leq \alpha_n(1-\alpha_n), \;\; \forall n \geq 1$ then from the inequality (16), we obtain

$$\begin{aligned} ||x_{n+1}-x'||^2 + \beta_n^2||x_n - T(x_n,x_n)||^2 &\leq (2\beta_n^2 + \alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n) - 2\beta_n^2)||x_n - x'|| \\ &= ||x_n - x'|| \end{aligned} \tag{20}$$

By the Cauchy-Schwarz inequality,

$$\langle T(x_n,x_n)-x',x_n-x'\rangle \leq ||T(x_n,x_n)-x'||\,||x_n-x'|| \leq ||x_n-x'||.$$

By (20), we get

$$\beta_n^2||x_n-T(x_n,x_n)||^2 \leq ||x_n-x'||^2-||x_{n+1}-x'||^2, \ \ \forall n\geq 1. \tag{21}$$

Thus $\{||x_n-x'||\}$ is a decreasing sequence, hence it is convergent. By the inequality (20), we get

$$\begin{aligned} &0 \leq ||x_n-T(x_n,x_n)||^2 \\ &\leq \frac{1}{\beta_n^2}(||x_n-x'||^2-||x_{n+1}-x'||^2)||x_n-x'||^2-||x_{n+1}-x'||^2, \ \ \forall n\geq 1, \end{aligned} \tag{22}$$

Take $n\to\infty$, we have $||x_n-T(x_n,x_n)||\to 0$. By demicompactness of T that there exist a converges subsequence $\{x_{n_k}\}$ of x_n in K, say $x_{n_k}\to w$. Since T is coupled-nonexpansive, we get T is continuous, and then $T(x_{n_k},x_{n_k})\to T(w,w)$. Since $||x_n-T(x_n,x_n)||\to 0$, we have $x_{n_k}-T(x_{n_k},x_{n_k})\to w-Tw,w)$, which shows that (w,w) is a coupled fixed point of T. Using (20), with $x'=w$, we deduce that the sequence of nonnegative real numbers $\{x_n-w\}$ is non-increasing, so convergent. Since its subsequence $\{x_{n_k}-w\}$ converges to 0, it follows that the sequence $\{x_n-w\}$ itself converges to 0, Therefore $\{(x_n,x_n)\}$ converges strongly to (w,w) as $n\to\infty$. □

Example 13. Let $\mathbb{R}$ be a real number. Defined $\langle x,y\rangle = xy$, and $|x|^2=\langle x,x\rangle$, for every $x,y\in\mathbb{R}$ and $T:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by

$$T(x,y)=\frac{x-y}{2},$$

for all $x,y\in\mathbb{R}$. Indeed, we see that consider, T satisfies (1) and is demicompact. Hence, all the assumptions of Theorem 8 are satisfied. It is easy to see that T possesses a unique coupled fixed point, $(0,0)$, and the Mann-type iteration (15) yields the sequence $x_n=(1-\alpha_n)^n x_0$, $n\geq 0$, where $\alpha_n=\frac{2n+2}{2n}\subseteq[0,1]$. Since $\lim_{n\to\infty}\alpha_n=\frac{1}{2}$, it follows that $(x_n,x_n)\to(0,0)$ as $n\to\infty$, for any $x_0\in K$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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