Thai Journal of Mathematics : 89-101
Special Issue : Annual Meeting in Mathematics 2019
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

The Rectangular Quasi-Metric Space and Common Fixed Point Theorem for ψ-Contraction and ψ-Kannan Mappings

Wongvisarut Khuangsatung[†], Saowapak Chan-iam[‡], Patchara Muangkarn[‡] and Cholatis Suanoom[‡,§,1]

[†]Department of Mathematics and Computer Science,
Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi, Pathumthani, 12110, Thailand,
e-mail : wongvisarutk@9rmutt.ac.th (W. Khuangsatung)
[‡]Program of Mathematics, Faculty of Science and Technology,
Kamphaengphet Rajabhat University, Kamphaengphet 62000, Thailand,
e-mail : booklnw23@hotmail.com (S. Chan-iam)
putchara_31@windowslive.com (P. Muangkarn)
[§]Science and Applied Science center, Faculty of Science and Technology,
Kamphaengphet Rajabhat University, Kamphaengphet 62000, Thailand,
e-mail : cholatis.suanoom@gmail.com (C. Suanoom)

Abstract : In this work, we extend and improve rectangular metric spaces to rectangular quasi-metric spaces by using the concept of quasi-metric spaces. Next, we obtain fixed point theorems in rectangular quasi-metric spaces. Moreover, we present some examples to illustrate and support our results.

Keywords : fixed point; quasi-metric space; rectangular metric space; rectangular quasi-metric space.

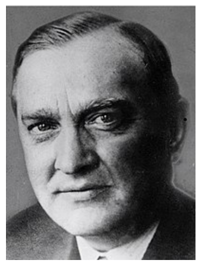
2010 Mathematics Subject Classification : 47H09; 47H10.

---

[1] Corresponding author.

1 Introduction and Preliminaries

In 1922, Banach [1] proved a fixed point theorem for metric spaces, which later on came to be known as the famous "Banach contraction principle".



Stefan Banach

Let (X, d) be a metric space. Then a map $T : X \to X$ is called a *contraction mapping* on X, if there exists $q \in [0, 1)$ such that

$$d(T(x), T(y)) \leq qd(x, y)$$

for all x, y in X. If (X, d) is a complete metric space with a contraction mapping $T : X \to X$, then T admits a unique fixed-point $x*$ in X. Furthermore, We can to find $x*$ as follows: We start x_0 in X and define a sequence x_n by $x_n = T(x_{n-1})$, then $x_n \to x*$. After that, we well-known to Banach Fixed Point Theorem.

Now, we recall definition of metric spaces was introduced by Frechet [2] as follows :

Definition 1.1. Let X be a non-empty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies :

$(MS1)$ $d(x, y) = 0$ *if and only if* $x = y$,
$(MS2)$ $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
$(MS3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfying (MS1)-(MS3), then d is called a metric on X and (X, d) is called a metric space.

Example 1.2. Let $X = \mathbb{R}$ and defined $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}$. Then (X, d) is metric spaces.

In 1931, Wilson [3] introduced quasi-metric spaces as follows :

Definition 1.3. Let X be a nonempty set. Suppose that the mapping $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

$(QS1)$ $d(x, y) = 0$ *if and only if* $x = y$;
$(QS2)$ $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

If d satisfies conditions (QS1) and (QS2), then d is called a quasi-metric on X and (X, d) is called a quasi-metric space.

Example 1.4. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}\}$ and $B = [1, 5]$. Define the generalized metric d on X as follows :
$d(\frac{1}{2}, \frac{1}{3}) = 0.3$, $d(\frac{1}{3}, \frac{1}{2}) = 0.2$, $d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = 0$, and $d(x, y) = |x - y|$.
If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$,
then (X, d) is a quasi-metric space, but it is not metric space.

In 2000, Branciari [4] introduced rectangular metric spaces as follows :

Definition 1.5. Let X be a none-mpty set and Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies:

$(RMS1)$ $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
$(RMS2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(RMS3)$ $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y, z \in X$ and all distinct point $u, v \in X \backslash \{x, y\}$.

Then d is called a rectangular metric on X and (X, d) is called a rectangular metric space.

Example 1.6 ([5])**.** Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1, 2]$. Define the generalized metric d on X as follows :

$$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3, \qquad d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.2,$$

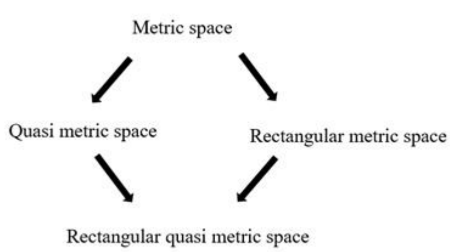
$$d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6, \qquad d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0$$

and $d(x, y) = |x - y|$ if $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$.
It is clear that d does not satisfy the triangle inequality in metric space,

$$0.6 = d(\frac{1}{2}, \frac{1}{4}) \geq d(\frac{1}{2}, \frac{1}{3}) + d(\frac{1}{3}, \frac{1}{4}) = 0.5.$$

Then d is a rectangular metric, but it is not a metric.

In this work, we extend and improve rectangular metric spaces to rectangular quasi-metric spaces by using the concept of quasi-metric spaces. Next, we obtain fixed point theorems in rectangular quasi-metric spaces. Moreover, we present some examples to illustrate and support our results.i.e,



2 Main Results

In this section, we introduce rectangular quasi-metric spaces and prove fixed point theorems. Likewise, we present some examples to illustrate and support our results.

Definition 2.1. Let X be a non-empty set and Suppose that the mappings $d : X \times X \longrightarrow [0, \infty)$ satisfies :
(RQMS1) $d(x, y) = 0$ if and only if $x = y$;
(RQMS2) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct points $u, v \in X \backslash \{x, y\}$.
Then d is called a rectangular quasi-metric on X and (X, d) is called a rectangular quasi-metric space.

Example 2.2. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1, 2]$. Define the generalized metric d on X as follows :

$$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3, \qquad d(\frac{1}{3}, \frac{1}{2}) = d(\frac{1}{5}, \frac{1}{4}) = 0.1,$$
$$d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6, \qquad d(\frac{1}{4}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{5}) = 0.4,$$
$$d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.2, \qquad d(\frac{1}{5}, \frac{1}{2}) = d(\frac{1}{4}, \frac{1}{3}) = 0.5,$$
$$d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0,$$

and

$$d(x, y) = |x - y| \text{ if } x, y \in B \text{ or } x \in A, y \in B \text{ or } x \in B, y \in A \ .$$

It is clear that d does not satisfy the triangle inequality A

$$0.6 = d(\frac{1}{2}, \frac{1}{4}) \geq d(\frac{1}{2}, \frac{1}{3}) + d(\frac{1}{3}, \frac{1}{4}) = 0.5.$$

We see that d is not a rectangular metrics, because $d(\frac{1}{2}, \frac{1}{4}) \neq d(\frac{1}{4}, \frac{1}{2})$. So d is a rectangular quasi-metric. Indeed,
(RMQ1)
($\Rightarrow$) Suppose that $d(x, y) = 0$.
Case(I) If $x, y \in A$, then $x = y$.
Case(II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$ then $d(x, y) = |x - y| = 0$, so $x = y$.
($\Leftarrow$) Suppose that $x = y$.
To show that $d(x, y) = 0$. we prove by two case.
Case(I) If $x, y \in A$ then $d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0$.
Case(II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$ then $x - y = 0$.
Thus $d(x, y) = |x - y| = 0$.

This is a proof of (RQM1)
(RQM2)
Case (I) If $x, y \in A$ then
$d(x,y) = d(\frac{1}{2}, \frac{1}{3}) = 0.3 \leq d(\frac{1}{2}, u) + d(u,v) + d(v, \frac{1}{3})$ when $u, v \in \{\frac{1}{4}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{3}, \frac{1}{2}) = 0.1 \leq d(\frac{1}{3}, u) + d(u,v) + d(v, \frac{1}{2})$ when $u, v \in \{\frac{1}{4}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{3}, \frac{1}{4}) = 0.2 \leq d(\frac{1}{3}, u) + d(u,v) + d(v, \frac{1}{4})$ when $u, v \in \{\frac{1}{2}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{4}, \frac{1}{3}) = 0.2 \leq d(\frac{1}{4}, u) + d(u,v) + d(v, \frac{1}{3})$ when $u, v \in \{\frac{1}{2}, \frac{1}{5}\}$
$d(x,y) = d(\frac{1}{4}, \frac{1}{5}) = 0.3 \leq d(\frac{1}{4}, u) + d(u,v) + d(v, \frac{1}{5})$ when $u, v \in \{\frac{1}{2}, \frac{1}{3}\}$
$d(x,y) = d(\frac{1}{5}, \frac{1}{4}) = 0.1 \leq d(\frac{1}{5}, u) + d(u,v) + d(v, \frac{1}{4})$ when $u, v \in \{\frac{1}{2}, \frac{1}{3}\}$.
Case (II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$, then

$$\begin{aligned} d(x,y) &= |x-y| \\ &\leq |x-u| + |u-y| \\ &\leq |x-u| + |u-v| + |v-y|, \end{aligned}$$

for all distinct points $u, v \in X \backslash \{x, y\}$.

Now, we introduce a definition of a convergent, cauchy, complete rectangular quasi-metric space as follows : For any $x \in X$, we define the open ball with centre x and radius $r > 0$ by

$$B_r(x); = \{y \in X | \max\{d(x,y), d(y,x)\} < r\}.$$

Definition 2.3. Let (X, d) be a rectangular quasi-metric space and let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ in X is called convergence to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n)$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

(b)The sequence $\{x_n\}$ in X is called cauchy sequence in (X, d) if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 = \lim_{n\to\infty} d(x_{n+p}, x_n)$, for all $p > 0$.

(c) (X, d) is called complete rectangular quasi metric space if every Cauchy sequence in X convergence to some $x \in X$.

Next, we present main theorems as follows :

Theorem 2.4. *Let (X, d) be a complete rectangular quasi-metric space. A mapping $g : X \to X$ satisfies:*

$$d(g(x), g(y)) \leq \psi(d(x,y)), \tag{2.1}$$

for all $x, y \in X$, where
(i) $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing and continuous functions,
(ii) $\sum_{i=n}^{\infty} \psi^i(t) + \psi^m(t^*) < \infty$ for $t, t^* > 0$ and for $m, n \in \mathbb{N}$,
(iii) $\psi(0) = 0$ and $\psi(t) < t$ for $0 < t$.
Then g has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitraty. We define a sequence $\{x_n\}$ by $x_{n+1} = gx_n$ for all $n = 0, 1, 2, \ldots$,. We will show that $\{x_n\}$ is Cauchy sequence, i.e., $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 = \lim_{n\to\infty} d(x_{n+p}, x_n)$ for all $p > 0$. If $x_n = x_{n+1}$ then x_n is fixed point of g, i.e., $x_n = gx_n$. So, suppose that $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \ldots$.
We consider

$$\begin{aligned} e_n := d(x_n, x_{n+1}) &= d(gx_{n-1}, gx_n) \\ &\leq \psi(d(x_{n-1}, x_n)) \\ &= \psi(d(gx_{n-2}, gx_{n-1})) \\ &\leq \psi^2(d(x_{n-2}, x_{n-1})) \\ &= \psi^2(d(gx_{n-3}, gx_{n-2})) \\ &\vdots \\ &\leq \psi^n(d(x_0, x_1)) \\ &= \psi^n(e_0), \end{aligned} \tag{2.2}$$

and,

$$\begin{aligned} l_n := d(x_{n+1}, x_n) &= d(gx_n, gx_{n-1}) \\ &\leq \psi(d(x_n, x_{n-1})) \\ &= \psi(d(gx_{n-1}, gx_{n-2})) \\ &\leq \psi^2(d(x_{n-1}, x_{n-2})) \\ &= \psi^2(d(gx_{n-2}, gx_{n-3})) \\ &\vdots \\ &\leq \psi^n(d(x_1, x_0)) \\ &= \psi^n(l_0). \end{aligned} \tag{2.3}$$

Since (2.2) and (2.3), we have $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$ and $d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0))$.
We consider

$$\begin{aligned} e_n^* := d(x, x_{n+2}) &= d(gx_{n-1}, gx_{n+1}) \\ &\leq \psi(d(x_{n-1}, x_{n+1})) \\ &= \psi(d(gx_{n-2}, gx_n)) \\ &\leq \psi^2(d(x_{n-2}, x_n)) \\ &\vdots \\ &\leq \psi^n(d(x_0, x_2)) \\ &= \psi^n(e_0^*), \end{aligned} \tag{2.4}$$

and,

$$\begin{aligned} l_n^* := \ d(x_{n+2}, x_n) &= d(gx_{n+1}, gx_{n-1}) \\ &\leq \psi(d(x_{n+1}, x_{n-1})) \\ &= \psi(d(gx_n, gx_{n-2})) \\ &\leq \psi^2(d(x_n, x_{n-2})) \\ &\vdots \\ &\leq \psi^n(d(x_2, x_0)) \\ &= \psi^n(l_0^*). \end{aligned} \tag{2.5}$$

Now, if p is odd say $2m+1$ then we obtain that

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1}) \\ &\leq e_n + e_{n+1} + [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\ &\leq e_n + e_{n+1} + e_{n+2} + ... + e_{n+2m} \\ &\leq \psi^n(e_0) + \psi^{n+1}(e_0) + \psi^{n+2}(e_0) + ... + \psi^{n+2m}(e_0) \\ &= \sum_{i=n}^{n+2m} \psi^i(e_0) \leq \sum_{i=n}^{\infty} \psi^i(e_0) < \infty. \end{aligned} \tag{2.6}$$

If p is even say $2m$ then we obtain that

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m}) \\ &\leq e_n + e_{n+1} + [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\ &\leq e_n + e_{n+1} + e_{n+2} + ... + d(x_{n+2m-2}, x_{n+2m}) \\ &= e_n + e_{n+1} + ... + e_{n+2m-2}^* \\ &\leq \psi^n(e_0) + \psi^{n+1}(e_0) + ... + \psi^{n+2m-2}(e_0^*) \\ &= \sum_{i=n}^{n+2m-2} \psi^i(e_0) + \psi^{n+2m-n}(e_0^*) \\ &\leq \sum_{i=n}^{\infty} \psi^i(e_0) + \psi^{n+2m-n}(e_0^*) < \infty. \end{aligned} \tag{2.7}$$

Similarly, if p is odd say $2m+1$ then we get that

$$\begin{aligned} d(x_{n+2m+1},x_n) \leq & d(x_{n+2m+1},x_{n+2m}) + d(x_{n+2m},x_{n+2m-1}) + d(x_{n+2m-1},x_n) \\ \leq & l_{n+2m+1} + l_{n+2m} + [d(x_{n+2m-1},x_{n+2m-2}) \\ & + d(x_{n+2m-2},x_{n+2m-3}) + d(x_{n+2m-3},x_n)] \\ \leq & \psi^{n+2m+1}(l_0) + \psi^{n+2m}(l_0) + \ldots + \psi^{n-1}(l_0) \\ = & \sum_{i=n-1}^{n+2m+1} \psi^i(l_0) \leq \sum_{i=n-1}^{\infty} \psi^i(l_0) < \infty. \end{aligned} \tag{2.8}$$

Similarly, if p is even say $2m$ then we get that

$$\begin{aligned} d(x_{n+2m},x_n) \leq & d(x_{n+2m},x_{n+2m-1}) + d(x_{n+2m-1},x_{n+2m-2}) + d(x_{n+2m-2},x_n) \\ \leq & l_{n+2m} + l_{n+2m-1} + [d(x_{n+2m-2},x_{n+2m-3}) \\ & + d(x_{n+2m-3},x_{n+2m-4}) + d(x_{n+2m-4},d(x_n)] \\ \leq & \psi^{n+2m}(l_0) + \psi^{n+2m-2}(l_0) + \ldots + \psi^{n-2}(l_0^*) \\ = & \sum_{i=n-2}^{n+2m} \psi^i(l_0) + \psi^{n-2}(l_0^*) \\ \leq & \sum_{i=n-2}^{\infty} \psi^i(l_0) + \psi^{n-2}(l_0^*) < \infty \end{aligned} \tag{2.9}$$

It follows from (2.6), (2.7), (2.8) and (2.9) that $\lim_{n\to\infty} d(x_n,x_{n+p}) = 0 = \lim_{n\to\infty} d(x_{n+p},x_n)$ for all $p > 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X,d). By completeness of (X,d) there exists a $u \in X$ such that $\lim_{n\to\infty} x_n = u$. We will show that u is a fixed point of g. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,gu) &\leq d(u,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},gu) \\ &= d(u,x_n) + e_n + d(gx_n,gu) \\ &\leq d(u,x_n) + e_n + \psi(d(x_n,u)). \end{aligned} \tag{2.10}$$

And, we get that

$$\begin{aligned} d(gu,u) &\leq d(gu,x_{n+1}) + d(x_{n+1},x_n) + d(x_n,u) \\ &= d(gu,gx_n) + l_n + d(x_n,u) \\ &\leq \psi(d(u,x_n)) + l_n + d(x_n,u). \end{aligned} \tag{2.11}$$

Using (2.10) and (2.11) it follows that $d(u,gu) = 0 = d(gu,u)$. So $gu = u$. Thus u is a fixed point of g. For uniqueness, let v be another a fixed point of g. Then it follows that $d(u,v) = d(gu,gv) \leq \psi(d(u,v)) < d(u,v)$ and $d(v,u) = d(gv,gu) \leq \psi(d(v,u)) < d(v,u)$, which is a contradiction. Therefore, we must have $d(u,v) = 0 = d(v,u)$. So $u = v$. Thus u is a fixed point of g. □

Next, we obtain corollary by set $\psi(t) = \exists r(t), \forall t \in [0,\infty), r \in [0,1)$.

Corollary 2.1. Let (X, d) be a complete rectangular quasi-metric space.Suppose that $T : X \longrightarrow X$ $x, y \in X$

$$d(gx, gy) \leq rd(x, y)$$

for all $x, y \in X$ where $r \in [0, 1)$. Then g has a unique fixed point in X.

Example 2.5. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1, 2]$. Define the generalized metric d on X as follows :

$$d(\tfrac{1}{2}, \tfrac{1}{3}) = d(\tfrac{1}{4}, \tfrac{1}{5}) = 0.3, \qquad d(\tfrac{1}{3}, \tfrac{1}{2}) = d(\tfrac{1}{5}, \tfrac{1}{4}) = 0.1,$$
$$d(\tfrac{1}{2}, \tfrac{1}{4}) = d(\tfrac{1}{5}, \tfrac{1}{3}) = 0.6, \qquad d(\tfrac{1}{4}, \tfrac{1}{2}) = d(\tfrac{1}{3}, \tfrac{1}{5}) = 0.4,$$
$$d(\tfrac{1}{2}, \tfrac{1}{5}) = d(\tfrac{1}{3}, \tfrac{1}{4}) = 0.2, \qquad d(\tfrac{1}{5}, \tfrac{1}{2}) = d(\tfrac{1}{4}, \tfrac{1}{3}) = 0.5,$$
$$d(\tfrac{1}{2}, \tfrac{1}{2}) = d(\tfrac{1}{3}, \tfrac{1}{3}) = d(\tfrac{1}{4}, \tfrac{1}{4}) = d(\tfrac{1}{5}, \tfrac{1}{5}) = 0,$$

and

$$d(x, y) = |x - y| \text{ if } x, y \in B \text{ or } x \in A, y \in B \text{ or } x \in B, y \in A\ .$$

Then (X, d) is a complete rectangular quasi-metric space.

Next, let $g : X \longrightarrow X$ by

$$gx = \begin{cases} \frac{1}{5} & x \in A, \\ \frac{x}{6} & x \in B, \end{cases}$$

where $\psi(t) = \frac{t}{2};\ \ \forall t \in [0, \infty)$. Then g satisfy Theorem 2.4, and we see that $\frac{1}{5}$ is a fixed point of g. Indeed,
Case(I) If $x, y \in A$, then $d(gx, gy) = d(\frac{1}{5}, \frac{1}{5}) = 0 \leq \frac{d(x,y)}{2} = \psi(d(x, y))$.
Case (II) If $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$, then

$$\begin{aligned} d(gx, gy) &= |gx - gy| \\ &= |\frac{x}{6} - y|;\ (set\ x \in B) \\ &\leq \ \frac{1}{2}|x - y| \\ &= \frac{d(x, y)}{2} \\ &= \psi(d(x, y)). \end{aligned} \tag{2.12}$$

In 1982, Sessa [6] introduced a common fixed point theorem for a selfmapping of a complete metric space as follows :

Definition 2.6. Two self-mappings S and T of metric space (X, d) are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx), \qquad \forall x \in X.$$

It is clear that two commuting mappings are weakly commuting

In 1986, Jungck [7] introduced a compatible mappings and common fixed points as follows :

Definition 2.7. Let T and S be two self-mappings of a metric space (X, d). S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

It is easy to see that two compatible maps are weakly compatible.

In 2002, Aamri and El Moutawakil [8] defined a new property called the (E.A) property which generalizes the concept of non-compatible mappings and proved some common fixed point theorems.

Definition 2.8. Let S and T be two self-mappings of a rectangular quasi-metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Example 2.9. (1) Let $X = [0, +\infty]$.Define $T, S : X \longrightarrow X$ by

$$Tx = \frac{x^2}{4} \text{ and } Sx = \frac{3x^2}{4} \text{ , } \forall x \in X.$$

Consider the sequence $x_n = 1/n$. Clearly $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$. Then T and S satisfy (E.A).

(2) Let $X = [2, +\infty]$. Define $T, S : X \longrightarrow X$ by

$$Tx = x + 1 \text{ and } Sx = 2x + 1 \text{ , } \forall x \in X.$$

Suppose that property (E.A) hold,Then there exists a $\{x_n\}$ in X sequence satisfying

$$\lim_{n\to\infty} Tx = \lim_{n\to\infty} Sx = t, \qquad \text{for some } t \in X.$$

Therefore

$$\lim_{n\to\infty} x_n = t - 1 \text{ and } \lim_{n\to\infty} x_n = \frac{t-1}{2} \text{ .}$$

then $t = 1$, which is a contradiction $1 \notin X$. Hence T and S do not satisfy (E.A).

Theorem 2.2. Let S and T be two weakly compatible self-mappings of a rectangular quasi-metric spaces (X, d) such that

(i) T and S satisfy the property (E.A),

(ii) $d(Tx, Ty) < \max\{d(Sx, Sy) \frac{[d(Tx,Sx)+d(Ty,Sy)]}{2}, \frac{[d(Ty,Sx)+d(Tx,Sy)]}{2}\}$, $\forall x \neq y \in X$,

(iii) $TX \subset SX$,

(iv) SX or TX is complete subspace of X.

Then T and S have a unique common fixed point.

Proof. Since T and S satisfy the property (E.A), there exists a sequence $\{x_n\}$ in X satifying

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X.$$

Suppose that SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$. Also $\lim_{n\to\infty} Tx_n = Sa$. We show that $Ta = Sa$. Suppose that $Ta \neq Sa$. Condition (ii) implies

$$\begin{aligned} d(Ta, Tx_n) < \max\{&d(Sa, Sx_n), [d(Ta, Sa) + d(Tx_n, Sx_n)]/2. \\ &[d(Tx_n, Sa) + d(Ta, Sx_n)]/2\}. \end{aligned} \tag{2.13}$$

Letting $n \to +\infty$ yields

$$\begin{aligned} d(Ta, Sa) \leq \max\{&d(Sa, Sa), [d(Ta, Sa) + d(Sa, Sa)]/2, \\ &[d(Sa, Sa) + d(Ta, Sa)]/2\} \\ &\leq d(Ta, Sa)/2; \end{aligned} \tag{2.14}$$

a contradiction. Hence $Ta = Sa$.

Since T and S are a weakly compatible, $STa = TSa$ and $TTa = TSa = STa = SSa$.

Finally, we show that Ta is a common fixed point of T and S. Suppose that $Ta \neq TTa$. Then

$$\begin{aligned} d(Ta, TTa) < \max\{&d(Sa, STa), [d(Ta, Sa) + d(TTa, STa)]/2, \\ &[d(TTa, Sa) + d(Ta, STa)]/2\} \\ &< \max\{d(Ta, TTa), [d(TTa, Ta) + d(Ta, TTa)]/2\} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} d(TTa, Ta) < \max\{&d(STa, Sa), [d(TTa, STa) + d(Ta, Sa)]/2, \\ &[d(Ta, STa) + d(TTa, Sa)]/2\} \\ &< \max\{d(TTa, Ta), [d(Ta, TTa) + d(TTa, Ta)]/2\}. \end{aligned} \tag{2.16}$$

Since (2.15) and (2.16) we have
$d(Ta, TTa) + d(TTa, Ta) < \max\{d(Ta, TTa), [d(TTa, Ta) + d(Ta, TTa)]/2\} + \max\{d(TTa, Ta), [d(Ta, TTa) + d(TTa, Ta)]/2\} = d(Ta, TTa) + d(TTa, Ta)$, where $\max\{d(Ta, TTa), [d(TTa, Ta) + d(Ta, TTa)]/2\} \neq d(Ta, TTa)$ and $< \max\{d(TTa, Ta), [d(Ta, TTa) + d(TTa, Ta)]/2\} \neq d(TTa, Ta)$;
which is a contradiction. Hence $TTa = Ta$ and $STa = TTa = Ta$. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$. Uniquness of the common fixed point, suppose that a, b are distinct common fixed

point of S and T .

$$d(a,b) = d(Ta,Tb) < \max\{d(Sa,Sb), \frac{[d(Ta,Sa)+d(Tb,Sb)]}{2}, \frac{[d(Tb,Sa)+d(Ta,Sb)]}{2}\}, = \frac{d(Tb,Sa)+d(Ta,Sb)}{2} = \frac{d(b,a)+d(a,b)}{2} \tag{2.17}$$

and

$$d(b,a) = d(Tb,Ta) < \max\{d(Sb,Sa), \frac{[d(Tb,Sb)+d(Ta,Sa)]}{2}, \frac{[d(Ta,Sb)+d(Tb,Sa)]}{2}\}, = \frac{d(Ta,Sb)+d(Tb,Sa)}{2} = \frac{d(a,b)+d(b,a)}{2}. \tag{2.18}$$

Since (2.17) and (2.18) we get that $d(a,b)+d(b,a) < \frac{d(b,a)+d(a,b)}{2} + \frac{d(a,b)+d(b,a)}{2}$ □

Example 2.3. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$. Define the generalized metric d on X as follows :
$d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{5}) = 0.3,\quad d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{4}) = 0.2,$
$d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{3}) = 0.6,\quad d(\frac{1}{2}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{5}) = 0,$
such that $d(x,y) = d(y,x)$ and
$d(x,y) = |x-y|$ if $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$. Define $T, S : X \longrightarrow X$ by

$$Tx = \frac{3x}{4} \text{ and } Sx = \frac{x^2}{2}, \qquad \forall x \in X.$$

Then

(1) T and S satisfy the property (E.A) for the sequence $x_n = 1 + 1/n, n = 1, 2, ...,$

(2) S and T are weakly compatible,

(3) T and S satisfy for all $x \neq y$,

(4) $T1 = S1 = 1$.

Acknowledgements : I would like to thank the referees for his comments and suggestions on the manuscript. This work was supported by the National Research Council of Thailand and Mathematical Association of Thailand. The authors would like to thank the Research and Development Institute of Kamphaeng Phet Rajabhat University, which provides funding for research.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133–181.

[2] M. Frechet, Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo 22 (1906) 1–72.

[3] W.A. Wilson, On quasi-metric spaces. Amer. J. Math. 53 (1931) 675–684 .

[4] A. Branciari, A fixed point theorem of Banach-Caccippoli type on a class of generalised metric space, Publ. Math. Debrecen 57 (2000) 31–37.

[5] H. Aydi, Fixed point result on a class of generalized metric spaces, 2012.

[6] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. 32 (1982) 149–153.

[7] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986) 771–779 .

[8] M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions J. Math. Anal. Appl. 270 (2002) 181–188.

(Received 17 June 2019)
(Accepted 24 December 2019)