Graphs and Combinatorics (2019) 35:1619–1632
https://doi.org/10.1007/s00373-019-02109-z

ORIGINAL PAPER

Naturally ordered strong endomorphisms on graphs

Nirutt Pipattanajinda[1] · Yangkok Kim[2] · Srichan Arworn[3]

Received: 10 May 2019 / Revised: 6 October 2019 / Published online: 23 October 2019
© Springer Japan KK, part of Springer Nature 2019

Abstract
In this paper, we study the natural partial order $\leq$ on $SEnd(G)$, the strong endomorphism monoid of a finite graph G and characterize minimal elements and maximal elements of $(SEnd(G), \leq)$. Then we introduce the concept of connectedness on $(SEnd(G), \leq)$ by using the natural partial order $\leq$ and determine the number of connected components of $SEnd(G)$ under certain conditions.

Keywords Natural partial orders · Strong endomorphism monoid · Graph

Mathematics Subject Classification 20M10 · 06A06 · 05C25

1 Introduction

Endomorphism and strong endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to (strong) endomorphism monoids of graphs and many interesting results concerning graphs and their (strong) endomorphism monoids have been obtained. The studies of the graph monoid is mostly on Green's relations, regular monoid, completely regular

This work was supported by The Thailand Research Fund (TRF) and Office of the Higher Education Commission, Thailand.

✉ Yangkok Kim
ykkim@deu.ac.kr

Nirutt Pipattanajinda
nirutt.p@gmail.com

Srichan Arworn
srichan288@yahoo.com

[1] Program of Mathematics, Faculty of Sciences and Technology, Kamphaeng Phet Rajabhat University, Kamphaeng Phet, Thailand

[2] Department of Mathematics, Dongeui University, Busan 614-714, Republic of Korea

[3] Department of Mathematics, Faculty of Sciences, Chiang Mai University, Chiang Mai 50200, Thailand

monoid, orthodox monoid, inverse monoid, endotype, etc. See [8,12] for some results about strong endomorphism monoids of graph, and see [1,1–4,6,7,9–11,14–18], the results for endomorphism monoids of graphs.

First of all, we introduce brief conceptual and literature reviews of results related to our studies.

Let G be a (finite) graph with the non-empty finite vertex set $V(G)$ and the edge set $E(G)$. For $x \in V(G)$, we write $N(x)$ for the neighborhood of x in G, i.e., $N(x) = \{x' \in V(G) : \{x, x'\} \in E(G)\}$. A mapping f from $V(G)$ into itself is called an **endomorphism** if f preserves adjacency, i.e., for all $x, x' \in V(G)$ we have

$$\{x, x'\} \in E(G) \Rightarrow \{f(x), f(x')\} \in E(G).$$

A mapping f from $V(G)$ into itself is called a **strong endomorphism** if for all $x, x' \in V(G)$,

$$\{x, x'\} \in E(G) \Leftrightarrow \{f(x), f(x')\} \in E(G).$$

Furthermore, a bijective endomorphism is called an **automorphism.**

We use the following notations:

- $End(G)$ for the set of all endomorphisms of G,
- $SEnd(G)$ for the set of all strong endomorphisms of G, and
- $Aut(G)$ for the set of all automorphisms of G.

We note that $Aut(G)$ forms a group and $End(G)$ forms a monoid with the usual composition of mappings. Furthermore, $SEnd(G)$ forms a regular monoid see [8].

For a semigroup S, we let $\mathcal{E}(S)$ to be the set of all idempotents of S. If S is regular, then $(S, \leq)$ is a poset under the relation $\leq$ on S defined by:

$$a \leq b \Leftrightarrow a = eb = bf \text{ for some } e, f \in \mathcal{E}(S).$$

In [13] the above partial order was extended to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \Leftrightarrow a = xb = by \text{ and } a = ay,$$

for some $x, y \in S^1$ where S^1 denotes the set S if $(S, \cdot)$ has an identity, and the set S with an identity adjoined in the other case. This is now called the **natural partial order** on a semigroup S. In fact, when S is regular, this partial order equals the one defined above in terms of idempotents (see [13]).

In Sect. 2 we investigate the natural partial order $\leq$ on $SEnd(G)$, the strong endomorphism monoid of a graph G and characterize minimal elements and maximal elements of $(SEnd(G), \leq)$. Moreover, we determine the numbers of upper bounds and lower bounds of $f \in SEnd(G)$. In Sect. 3 we define the concept of connectedness on $(SEnd(G), \leq)$ by using the natural partial order $\leq$ and determine the number of connected components of $SEnd(G)$ under certain conditions.

2 Characterizations of the natural order on *SEnd*(*G*)

We now characterize the natural partial order $\leq$ on $SEnd(G)$, which is an analogous version of the natural partial order on T_X, the full transformation semigroup on a finite set X. We recall that the following characterization of the natural partial order on T_X has been proved. We also note that the kernel equivalence π_f defined by $f \in T_X$ is given by : $x\pi_f y$ if and only if $f(x) = f(y)$.

Lemma 2.1 [5] *If X is any set, then the following are equivalent for $(T_X, \leq)$:*

(i) $f \leq g$.
(ii) $f = \alpha g$ *for some* $\alpha \in \mathcal{E}(T_X)$ *and* $Im(f) \subseteq Im(g)$.
(iii) $f = g\beta$ *for some* $\beta \in \mathcal{E}(T_X)$ *and* $\pi_f \supseteq \pi_g$.
(iv) *(1)* $Im(f) \subseteq Im(g)$, *(2)* $\pi_f \supseteq \pi_g$, *(3)* $f(x) = g(x)$ *for any* $x \in X$ *with* $g(x) \in Im(f)$.

We define an equivalence relation $\sim$ on $V(G)$ by

$$x \sim x' \text{ if and only if } N(x) = N(x')$$

for $x, x' \in V(G)$. For each $x \in V(G)$, we write $\overline{x}$ for the equivalence class to which x belongs, and by $\overline{G}$ we denote the set of all equivalence classes of $V(G)$ by $\sim$. Then we have a graph $\overline{G}$ with the vertex set

$$V\left(\overline{G}\right) = \{\overline{x} : x \in V(G)\}$$

and

$$E\left(\overline{G}\right) = \{\{\overline{x}, \overline{y}\} : \{x, y\} \in E(G)\}.$$

From now on when there are precisely m distinct equivalence classes, $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_m}$ of $V(G)$, we then write $V(G) = \overline{x_1} \cup \overline{x_2} \cup \cdots \cup \overline{x_m}$ for short. We note that if G has no loops and $\{x, y\} \in E(G)$, then $\overline{x} \cup \overline{y}$ forms a complete bipartite induced subgraph, that is, $\{x', y'\} \in E(G)$ for all $x' \in \overline{x}$ and $y' \in \overline{y}$.

Proposition 2.2 *For $f \in SEnd(G)$ and $x, y \in V(G)$, $\overline{x} = \overline{y}$ if and only if $\overline{f(x)} = \overline{f(y)}$.*

Proof Suppose that $\overline{x} = \overline{y}$. Let $\alpha \in N(f(x))$. We note that $V(G)$ is finite and so there are integers m, n such that $0 \leq m < n$ and $f^m(\alpha) = f^n(\alpha)$. Hence,

$$\begin{aligned}\{\alpha, f(x)\} \in E(G) &\Leftrightarrow \{f^m(\alpha), f^{m+1}(x)\} \in E(G) \Leftrightarrow \{f^n(\alpha), f^{m+1}(x)\} \in E(G)\\ &\Leftrightarrow \{f^{n-m-1}(\alpha), x\} \in E(G) \Leftrightarrow \{f^{n-m-1}(\alpha), y\} \in E(G)\\ &\Leftrightarrow \{f^n(\alpha), f^{m+1}(y)\} \in E(G) \Leftrightarrow \{\alpha, f(y)\} \in E(G).\end{aligned}$$

Conversely, suppose that $\overline{f(x)} = \overline{f(y)}$. On the contrary, suppose that $\overline{x} \neq \overline{y}$. So $N(x) \neq N(y)$ and without loss of generality, we can suppose that there exists $a \in N(x) \setminus N(y)$. If $f(a) \in N(f(y))$, then $\{f(a), f(y)\} \in E(G)$ and so $\{a, y\} \in E(G)$, which means $a \in N(y)$, a contraction. □

We note that $SEnd(G)$ is a regular submonoid of $T_{V(G)}$, and hence, by Theorem 1.5 in [17], the natural partial order on $SEnd(G)$ is the restriction of the natural partial order on $T_{V(G)}$ to $SEnd(G)$. Hence the following result follows from Lemma 2.1.

Theorem 2.3 *If G is a finite graph, the following are equivalent for $(SEnd(G), \leq)$ where $\mathcal{E}(SEnd(G))$ denotes the set of all idempotents of $SEnd(G)$:*

(i) $f \leq g$.
(ii) $f = \alpha g$ *for some* $\alpha \in \mathcal{E}(SEnd(G))$ *and* $Im(f) \subseteq Im(g)$.
(iii) $f = g\beta$ *for some* $\beta \in \mathcal{E}(SEnd(G))$ *and* $\pi_f \supseteq \pi_g$.
(iv) *(1)* $Im(f) \subseteq Im(g)$, *(2)* $\pi_f \supseteq \pi_g$, *(3)* $f(x) = g(x)$ *for any* $x \in V(G)$ *with* $g(x) \in Im(f)$.

□

In particular, if G is a finite empty graph, then $SEnd(G)$ is actually T_G, the full transformation semigroup on G, which induces that Theorem 2.3 and Lemma 2.1 coincide.

From now on for every subset $X \subseteq V(G)$ and $f \in SEnd(G)$, by $f|_X$ we mean the restriction of f to X.

Lemma 2.4 *Let $f, g \in SEnd(G)$ such that $f \leq g$. Then $\overline{f(x)} = \overline{g(x)}$, for every $x \in V(G)$.*

Proof Since $f = \alpha g$, for some idempotent α of $SEnd(G)$, we have $\alpha f = \alpha g$ and then $\overline{\alpha(f(x))} = \overline{\alpha(g(x))}$. Thus by Proposition 2.2 we have $\overline{f(x)} = \overline{g(x)}$ for every $x \in V(G)$. □

Let $f, g \in SEnd(G)$ and $f < g$. If there is no $h \in SEnd(G)$ such that $f < h < g$, then g is called an *upper cover* of f, and f is called a *lower cover* of g.

Lemma 2.5 *Let $f, g \in SEnd(G)$ such that $f \leq g$. If $Im(f) = Im(g)$ or $\pi_f = \pi_g$, then $f = g$.*

Proof See Corollary 2.4 in [5]. □

Lemma 2.6 *Let $f, g \in SEnd(G)$ and $f < g$. Then g is an upper cover of f if and only if $|Im(g) \backslash Im(f)| = 1$.*

Proof If $f < g$ and g is not an upper cover of f, then $f < h < g$ for some $h \in SEnd(G)$. By Lemma 2.5, $Im(f) \subsetneq Im(h) \subsetneq Im(g)$ and so $|Im(g) \backslash Im(f)| > 1$. Suppose that a, b are two distinct elements in $Im(g) \backslash Im(f)$. We define an endomorphism $h : V(G) \to V(G)$ by

$$h(x) = \begin{cases} f(x) & \text{if } g(x) = a \\ g(x) & \text{otherwise} \end{cases}.$$

Suppose $\{h(s), h(t)\} \in E(G)$. If $g(s) = g(t) = a$, then $\{h(s), h(t)\} = \{f(s), f(t)\} \in E(G)$ and so $\{s, t\} \in E(G)$. We can apply the same argument for the case of $g(s) \neq a$ and $g(t) \neq a$. Without loss of generality we suppose that

$g(s) \neq a$ and $g(t) = a$, that is, $h(s) = g(s)$ and $h(t) = f(t)$. We note that $f < g$ and so $f(t) \in Im(f) \subset Im(g)$. Thus $f(t) = g(w) = f(w)$ for some $w \in V(G)$. Hence,

$$\begin{aligned}\{h(s), h(t)\} \in E(G) &\Rightarrow \{g(s), f(t)\} \in E(G) \Rightarrow \{g(s), g(w)\} \in E(G)\\ &\Rightarrow \{s, w\} \in E(G) \Rightarrow \{f(s), f(w)\} \in E(G) \Rightarrow \{f(s), f(t)\} \in E(G)\\ &\Rightarrow \{s, t\} \in E(G),\end{aligned}$$

which means that h is a strong endomorphism. We now claim that $f \leq h$. We note that (1) by the definition of h, we have $Im(f) \subset Im(h)$. (2) Let $h(s) = h(t)$. If $g(s) = g(t) = a$, then $h(s) = f(s) = h(t) = f(t)$. If $g(s) \neq a$ and $g(t) \neq a$, then $h(s) = g(s) = h(t) = g(t)$ and so $f(s) = f(t)$. Without loss of generality we suppose that $g(s) \neq a$ and $g(t) = a$, that is, $h(s) = g(s)$ and $h(t) = f(t)$. Thus, $g(s) = h(s) = h(t) = f(t) \in Im(f)$. Since $g(s) \in Im(f)$, we have $g(s) = f(s)$ and specially $f(s) = f(t)$. So $(s, t) \in \pi_h$ implies that $(s, t) \in \pi_f$. (3) Suppose that $h(x) \in Im(f)$. If $g(x) \neq a$, then $h(x) = g(x) \in Im(f)$ and so $f(x) = g(x) = h(x)$. If $g(x) = a$, then $h(x) = f(x)$. By Theorem 2.3 (iv), $f \leq h$. We can apply the similar argument for $h \leq g$. We note that $Im(f) \subsetneq Im(h) \subsetneq Im(g)$ for $a \in Im(g) \backslash Im(h)$ and $b \in Im(h) \backslash Im(f)$. Hence, $f < h < g$ and so g is not an upper cover of f. □

Definition 2.1 Let G be a graph and $N(s) = N(t)$. Then the function $\eta : V(G) \to V(G)$ defined by

$$\eta(x) = \begin{cases} t & \text{if } x = s \\ x & \text{otherwise} \end{cases}$$

is called a reducing endomorphism.

Definition 2.2 A graph is called a *seedling graph* if there is no pair of distinct vertices u and v with $N(u) = N(v)$.

For $f \in SEnd(G)$, $Im(f)$ can be regarded as a subgraph of G with $V(Im(f)) = \{f(x) : x \in V(G)\}$ and $E(Im(f)) = \{\{f(x), f(y)\} : \{x, y\} \in E(G)\}$.

Lemma 2.7 *Let $f \in SEnd(G)$. If $Im(f)$ is a seedling graph, then $|f(\overline{x})| = 1$ for all $x \in V(G)$.*

Proof On the contrary, suppose that $x_1, x_2 \in \overline{x}$ and $f(x_1) \neq f(x_2)$. We note that $\overline{x_1} = \overline{x_2}$ and so $\overline{f(x_1)} = \overline{f(x_2)}$. That is, $N(f(x_1)) = N(f(x_2))$ and hence $N(f(x_1)) \cap Im(f) = N(f(x_2)) \cap Im(f)$. Thus $Im(f)$ has a pair of distinct vertices with the same neighborhood sets, which is a contradiction. So $|f(\overline{x})| = 1$ for all $x \in V(G)$. □

Theorem 2.8 *A strong endomorphism of a graph is minimal if and only if its image is a seedling graph.*

Proof Let $f \in SEnd(G)$ such that $Im(f)$ a seedling graph. Let $h \in SEnd(G)$ and $h \leq f$. Then $Im(h) \subseteq Im(f)$. Let $x \in V(G)$. By Lemma 2.4 $N(h(x)) = N(f(x))$,

and hence $N(h(x)) \cap Im(f) = N(f(x)) \cap Im(f)$. Since $Im(f)$ is a seedling graph and $h(x), f(x) \in Im(f)$, we have $h(x) = f(x)$. Therefore $h = f$ and so f is minimal. For the converse we suppose that $f \in SEnd(G)$ and $Im(f)$ is not a seedling graph. Then there exist $s, t \in Im(f)$ such that $s \neq t$ and $N(s) \cap Im(f) = N(t) \cap Im(f)$. We let $\eta : V(G) \to V(G)$ be a reducing endomorphism defined by

$$\eta(x) = \begin{cases} t & \text{if } x = s \\ x & \text{otherwise} \end{cases}.$$

Then (1) $Im(\eta f) \subseteq Im(f)$, (2) if $f(x) = f(y)$, then $\eta f(x) = \eta f(y)$ and so $\pi_{\eta f} \supseteq \pi_f$, and (3) if $f(x) \in Im(\eta f)$, then $f(x) \in Im(\eta f) \backslash \{s\}$ and so $\eta f(x) = f(x)$. Therefore by Theorem 2.3, we have $\eta f < f$ and so f is not minimal. □

Definition 2.3 For a graph G and $f \in SEnd(G)$, $x \in Im(f)$ is called a multi-preimage vertex of f if $|f^{-1}(x)| \geq 2$, where $f^{-1}(x) = \{s \in V(G) | f(s) = x\}$.

Lemma 2.9 *Let $f, g \in SEnd(G)$ such that $f \leq g$ and let $s \in V(G)$. If $f(s)$ is not a multi-preimage vertex of f, then $g(s)$ is not a multi-preimage vertex of g and $g(s) = f(s)$.*

Proof If $g(s)$ is a multi-preimage vertex of g, then $g(s) = g(t)$ for some $t \neq s$. Since $f \leq g$ and $\pi_g \subseteq \pi_f$, we have $f(s) = f(t)$, a contradiction. Since $Im(f) \subset Im(g)$, there is $f(s) = g(t)$ for some $t \in V(G)$, and $f(s) = g(t) \in Im(f)$. Since $f \leq g$, $f(s) = g(t) = f(t)$. We note that $f(s)$ is not a multi-preimage vertex of f. Thus $s = t$ and $f(s) = g(s)$. □

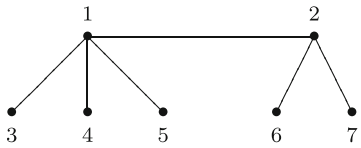
Theorem 2.10 *A strong endomorphism f of a graph G is maximal if and only if for every multi-preimage vertex x of f, $N(x) \cap Im(f) \neq N(y) \cap Im(f)$ for all $y \in V(G) \backslash Im(f)$.*

Proof On the contrary, we suppose there is a multi-preimage vertex x in $Im(f)$ with $N(x) \cap Im(f) = N(y) \cap Im(f)$ for some $y \in V(G) \backslash Im(f)$. Further, we let $f(s) = f(t) = x$, where s and t are two distinct vertices of G. We define the function $g : V(G) \to V(G)$ by

$$g(w) = \begin{cases} y & \text{if } w = s \\ f(w) & \text{otherwise} \end{cases}.$$

Since $f(s) = f(t) = x$ and f is a strong endomorphism, g is a strong endomorphism and $f < g$. On the contrary, we suppose that there exists $g \in SEnd(G)$ such that $f < g$. Then $\pi_f \supsetneq \pi_g$, which means that there exist $s, t \in V(G)$ such that $f(s) = f(t)$ and $g(s) \neq g(t)$. Thus $f(s)$ is a multi-preimage vertex of f. If $g(s), g(t) \in Im(f)$, then, by Theorem 2.3(iv), $g(s) = f(s)$ and $g(t) = f(t)$, whence $g(s) = g(t)$, a contradiction. Therefore $g(s) \notin Im(f)$ or $g(t) \notin Im(f)$. Since by Lemma 2.4, $N(f(s)) = N(g(s))$ and $N(f(t)) = N(g(t))$, and so $N(f(s)) \cap Im(f) = N(g(s)) \cap Im(f)$ and $N(f(t)) \cap Im(f) = N(g(t)) \cap Im(f)$, we get a contradiction and so f is maximal. □

Fig. 1 The graph G



It is clear that a bijective strong endomorphism (a graph automorphism) is maximal. We consider a graph $G = (V(G), E(G))$ where $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}\}$, and a strong endomorphisms:

$$f = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,1\,5\,5\,3 \end{pmatrix}.$$

Then f is not bijective, but it is maximal and minimal.

In the following we determine the numbers of upper covers and lower covers of an element of $SEnd(G)$.

We consider a graph G depicted in Fig. 1 and $f, g \in SEnd(G)$;

$$f = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,3\,3\,3\,6\,6 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,3\,4\,5\,6\,7 \end{pmatrix}.$$

We determine the number of upper covers of f, using Lemma 2.6. We write $V(G) = \overline{3} \cup \overline{6} \cup \{1, 2\}$, where $|\overline{3}| > 1$, $|\overline{6}| > 1$. Moreover, $f(\overline{3}) \subset \overline{3}$ and $f(\overline{6}) \subset \overline{6}$. Let $h \in SEnd(G)$ be an upper cover of f. Then by Lemma 2.6 $Im(h) \backslash Im(f) = \{a\}$ for some $a \in V(G)$. If $a = 4$, (or 5) then an upper cover of f is of the form

$$h = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,\epsilon_1\,\epsilon_2\,\epsilon_3\,6\,6 \end{pmatrix}$$

where $\{\epsilon_1, \epsilon_2, \epsilon_3\} = \{3, 4\}$ (or $\{\epsilon_1, \epsilon_2, \epsilon_3\} = \{3, 5\}$) respectively. Now there are $2^3 - 2$ upper covers of f in case $a = 4$ or $a = 5$. Thus there exist $2 \cdot (2^3 - 2)$ upper covers of f where we note the factor 2 is $|\overline{3} - f(\overline{3})|$. Similarly we have $2^2 - 2$ upper covers of f for $a = 7$. Hence, the number of all upper cover of f is

$$2 \cdot (2^3 - 2) + 2^2 - 2 = 14.$$

Again we use Lemma 2.6 to determine the number of lower covers of g. We write $V(G) = \overline{3} \cup \overline{6} \cup \{1, 2\}$, where $|\overline{3}| > 1$, $|\overline{6}| > 1$. Moreover, $g(\overline{3}) \subset \overline{3}$ and $g(\overline{6}) \subset \overline{6}$. Let $h < g$ and $Im(g) \backslash Im(h) = \{a\}$ for some $a \in V(G)$. If $a = 3$, then h is of the form

$$h = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,\epsilon\,4\,5\,6\,7 \end{pmatrix}$$

where $\epsilon = 4, 5$. The same argument applies for $a = 4, 5$. Similarly if $a = 6$ or 7, then the lower cover of g is

$$h = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,3\,4\,5\,7\,7 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,3\,4\,5\,6\,6 \end{pmatrix}.$$

Hence, the number of all lower covers of g is

$$3 \cdot 2 + 2 \cdot 1 = 8.$$

Based on the above example, we determine the number of upper covers of $f \in SEnd(G)$. Let $V(G) = \overline{x_1} \cup \cdots \cup \overline{x_s} \cup \overline{y_1} \cup \cdots \cup \overline{y_t}$, where $|\overline{x_i}| = 1$ and $|\overline{y_i}| > 1$. We suppose that h is an upper cover. Then $Im(h) \backslash Im(f) = \{a\}$ for some $a \in V(G)$. We note that $f|_{\overline{x_i}} = h|_{\overline{x_i}}$. Moreover, $f|_X = h|_X$ if $f(X) \subset \overline{x_i}$ or $f(X) = \overline{y_i}$ for some i. Hence, we only consider the case of $f(\overline{y_i}) \subsetneq \overline{f(y_i)}$ and $a \in \overline{f(y_i)}$. For the simplicity of notations, we let $\overline{z_1}, \ldots, \overline{z_m}$ be all equivalence classes such that $f(\overline{z_i}) \subsetneq \overline{f(z_i)}$, $|\overline{z_i}| > 1$, and $|\overline{f(z_i)}| > 1$. For each $\overline{z_i}$, we can write

$$\overline{z_i} = \Lambda_{i_1} \cup \cdots \cup \Lambda_{i_{s(i)}} \cup \{\beta_{i_1}, \ldots, \beta_{i_{t(i)}}\}$$

where $|\Lambda_{i_k}| = \ell_{i_k} > 1$, $f(\Lambda_{i_k}) = \{c_{i_k}\}$ and $f(\beta_{i_k}) = \{d_{i_k}\}$, and all constants c_{i_k} and d_{i_k} are distinct. We note that $a \in \overline{f(z_i)} \backslash f(\overline{z_i})$ and so there are p_i choices for a where $p_i = \left|\overline{f(z_i)} \backslash f(\overline{z_i})\right|$. Hence, if $a \in f(\Lambda_{i_k})$, there are $p_i(2^{\ell_{i_k}} - 2)$ upper covers. Thus $\sum_{k=1}^{s(i)} p_i(2^{\ell_{i_k}} - 2)$ upper covers exist for the case of $a \in \overline{f(z_i)} \backslash f(\overline{z_i})$. The number of all upper covers of f is

$$\sum_{i=1}^{m} \sum_{k=1}^{s(i)} p_i(2^{\ell_{i_k}} - 2).$$

Similarly we now determine the number of lower covers of $f \in SEnd(G)$. Let $V(G) = \overline{x_1} \cup \cdots \cup \overline{x_s} \cup \overline{y_1} \cup \cdots \cup \overline{y_t}$, where $|\overline{x_i}| = 1$ and $|\overline{y_i}| > 1$. We suppose that h is a lower cover. Then $Im(f) \backslash Im(h) = \{a\}$ for some $a \in V(G)$. We note that $f|_{\overline{x_i}} = h|_{\overline{x_i}}$. Moreover, $f|_X = h|_X$ if $f(X) \subset \overline{x_i}$ or $f(X) = \overline{y_i}$ for some i. Hence, we only consider the case of $f(\overline{y_i}) \subseteq \overline{f(y_i)}$ and $|f(\overline{y_i})| > 1$ and $a \in f(\overline{y_i})$. For the simplicity of notations, we let $\overline{z_1}, \ldots, \overline{z_m}$ be all equivalence classes such that $f(\overline{z_i}) \subsetneq \overline{f(z_i)}$, $|\overline{z_i}| > 1$, $|\overline{f(z_i)}| > 1$, and $|f(\overline{z_i})| > 1$. For each $\overline{z_i}$, we can write

$$\overline{z_i} = \Lambda_{i_1} \cup \cdots \cup \Lambda_{i_{s(i)}} \cup \{\beta_{i_1}, \ldots, \beta_{i_{t(i)}}\}$$

where $|\Lambda_{i_k}| = \ell_{i_k} > 1$, $f(\Lambda_{i_k}) = \{c_{i_k}\}$ and $f(\beta_{i_k}) = \{d_{i_k}\}$, and all constants c_{i_1} and d_{i_1} are distinct. If $a = c_{i_k}$, then $h(\Lambda_{i_k}) \in \{c_{i_1}, \ldots, c_{i_{s(i)}}, d_{i_1}, \ldots, d_{i_{t(i)}}\} \backslash \{c_{i_k}\}$. We note that $|f(\Lambda_{i_k})| = 1$ implies $|h(\Lambda_{i_k})| = 1$. There are $(s(i) + t(i) - 1)$ choices for $h(\Lambda_{i_k})$, independent of k. Hence if $a \in f(\Lambda_{i_k}) = \{c_{i_k}\}$, there are $s(i) + t(i) - 1$ lower covers. Similarly if $a \in f(\beta_{i_k}) = \{d_{i_k}\}$, then $h(\beta_{i_k}) \in \{c_{i_1}, \ldots, c_{i_{s(i)}}, d_{i_1}, \ldots, d_{i_{t(i)}}\} \backslash \{d_{i_k}\}$.

For each element β_{i_k}, there are $(s(i)+t(i)-1)$ choices for $h(\beta_{i_k})$, independent of k, and so $s(i)+t(i)-1$ lower covers. Thus if $a \in f(\overline{z_1})$, there are $(s(i)+t(i))\cdot(s(i)+t(i)-1)$ lower covers of f. Hence, the number of all lower covers of f is

$$\sum_{i=1}^{m}(s(i)+t(i))\cdot(s(i)+t(i)-1).$$

3 The number of connected components in $SEnd(G)$

In this section, we introduce the connectedness in $SEnd(G)$ and determine the number of connected components in $SEnd(G)$. We note that for every graph G, $SEnd(\overline{G}) = Aut(\overline{G})$.

Definition 3.1 Two elements f, g in $SEnd(G)$ are connected if there is a finite sequence of elements $f_1, \ldots, f_m$ in $SEnd(G)$ such that $f = f_1$, $f_m = g$, and $f_i \leq f_{i+1}$ or $f_{i+1} \leq f_i$ for $i = 1, \ldots, m-1$. A subset S of $SEnd(G)$ is said to be connected if any two distinct elements f, g in S are connected. The maximal connected subset is called a connected component.

For $\mu \in Aut(\overline{G})$, we define

$$X_\mu = \{f \in SEnd(G) : \overline{f(x)} = \mu(\overline{x}) \text{ for all } x \in V(G)\}.$$

We show $X_\mu \neq \emptyset$ for $\mu \in Aut(\overline{G})$. Let $\widehat{\eta}$ be a minimal endomorphism such that $\widehat{\eta} \leq 1$, an identity endomorphism of G. Then $Im(\widehat{\eta})$ is a seedling graph and so $Im(\widehat{\eta})$ is isomorphic to $\overline{G}$. Let $\pi : Im(\widehat{\eta}) \to \overline{G}$ be an isomorphism and $\rho = \pi^{-1}\mu\pi\widehat{\eta}$. Then $\rho : V(G) \to V(G)$ is a strong endomorphism. Moreover, $\rho(x) = \pi^{-1}\mu\pi\widehat{\eta}(x)$ and so $\pi\rho(x) = \mu\pi\widehat{\eta}(x)$, for all $x \in V(G)$, equivalently, $\overline{\rho(x)} = \mu(\widehat{\eta}(x)) = \mu(\overline{x})$, which means $\rho \in X_\mu$. For $f \in SEnd(G)$, we define $\mu : \overline{G} \to \overline{G}$ by $\mu(\overline{x}) = \overline{f(x)}$. If $\overline{x} = \overline{y}$, then $\overline{f(x)} = \overline{f(y)}$ and so μ is well-defined. Moreover,

$$\begin{aligned}&\{f(x), f(y)\} \in E(G) \Leftrightarrow \{x, y\} \in E(G) \Leftrightarrow \{\overline{x}, \overline{y}\} \in E(\overline{G})\\ &\quad\Leftrightarrow \{\overline{f(x)} = \overline{f(y)}\} \in E(\overline{G})\end{aligned}$$

which means $\mu \in Aut(\overline{G})$ and $SEnd(G) = \cup_{\mu\in Aut(\overline{G})}X_\mu$. Moreover, if $f \in X_\mu \cap X_\rho$ and $\mu \neq \rho$, then $\overline{f(x)} = \mu(\overline{x}) = \rho(\overline{x})$ for all $x \in V(G)$, a contradiction. Thus if $Aut(\overline{G}) = \{\mu_1, \mu_2, \ldots, \mu_t\}$, then $X_{\mu_1}, X_{\mu_2}, \ldots, X_{\mu_t}$ forms a partition of $SEnd(G)$. We note that for two distinct $\mu, \rho \in Aut(\overline{G})$, if $f \in X_\mu$ and $g \in X_\rho$, then $\overline{f(x)} \neq \overline{g(x)}$ for some $x \in V(G)$. By Lemma 2.4, $f \not\leq g$ and $g \not\leq f$, which means that an element in X_μ and an element in X_ρ are not comparable. If $f \in X_\mu$ and $g \in X_\rho$ is connected, there is a finite sequence of elements $f_1, f_2, \ldots, f_m$ in $SEnd(G)$ such that $f = f_1, f_2, \ldots, f_m = g$, and $f_i \leq f_{i+1}$ or $f_{i+1} \leq f_i$ for $i = 1, \ldots, m-1$. Let ℓ be the smallest integer such that $f_{\ell+1} \notin X_\mu$. Then f_ℓ and $f_{\ell+1}$ are not comparable, a contradiction. Hence, for two distinct $\mu, \rho \in Aut(\overline{G})$, an element in X_μ and an element in X_ρ are not connected. We now have the following.

Fig. 2 The graphs G and $\overline{G}$

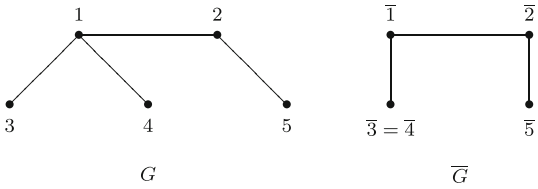
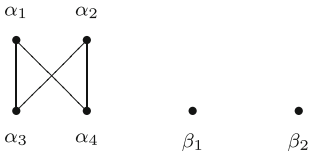


Fig. 3 Three connected components in $SEnd(G)$



Lemma 3.1 *If $|Aut(\overline{G})| = k$, there are at least k connected components in $SEnd(G)$.* □

We consider a graph $G = (V(G), E(G))$ where $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 5\}\}$, and $\overline{G}$ (See Fig. 2, where $\overline{1} = \{1\}$, $\overline{2} = \{2\}$, $\overline{3} = \overline{4} = \{3, 4\}$ and $\overline{5} = \{5\}$).

Then $SEnd(G) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2\}$, where

$$\alpha_1 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,3\,4\,5 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,4\,3\,5 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,3\,3\,5 \end{pmatrix},$$
$$\alpha_4 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,4\,4\,5 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,1\,5\,5\,3 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,1\,5\,5\,4 \end{pmatrix}$$

and

$$Aut(\overline{G}) = \{\alpha = \begin{pmatrix} \overline{1}\,\overline{2}\,\overline{3}\,\overline{5} \\ \overline{1}\,\overline{2}\,\overline{3}\,\overline{5} \end{pmatrix}, \quad \beta = \begin{pmatrix} \overline{1}\,\overline{2}\,\overline{3}\,\overline{5} \\ \overline{2}\,\overline{1}\,\overline{5}\,\overline{3} \end{pmatrix}\}.$$

Moreover, $X_\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $X_\beta = \{\beta_1, \beta_2\}$. We note that X_α is connected. However, $\beta_1 \not\sim \beta_2$ and $\beta_2 \not\sim \beta_1$. Hence $SEnd(G)$ has 3 connected components depicted in Fig. 3.

Lemma 3.2 *Let $h, f \in SEnd(G)$. If $Im(h)$ is a seedling graph and $h(x) \in Im(f|_{\overline{x}})$ for all $x \in V(G)$, then $h \leq f$.*

Proof We prove the assertion by Theorem 2.3. (1) Clearly we have $Im(h) \subset Im(f)$. (2) If $f(x) = f(y)$, then $\overline{f(x)} = \overline{f(y)}$ and so $\overline{x} = \overline{y}$. Since $Im(h)$ is a seedling graph, $h(x) = h(y)$ and so $\pi_f \subseteq \pi_h$. (3) Let $x \in V(G)$ such that $f(x) \in Im(h)$. Then $f(x) = h(y) \in Im(f|_{\overline{y}})$ for some $y \in V(G)$, and so $f(x) = f(y')$ for some $y' \in \overline{y}$. Hence, $\overline{x} = \overline{y'} = \overline{y}$ and $h(x) = h(y)$ for $Im(h)$ is a seedling graph, which implies $f(x) = h(x)$, and $h \leq f$ by Theorem 2.3 (iv). □

Theorem 3.3 *For $f, g \in SEnd(G)$, there is a lower bound of f and g if and only if $Im(f|_{\overline{x}}) \cap Im(g|_{\overline{x}}) \neq \emptyset$ for all $x \in V(G)$.*

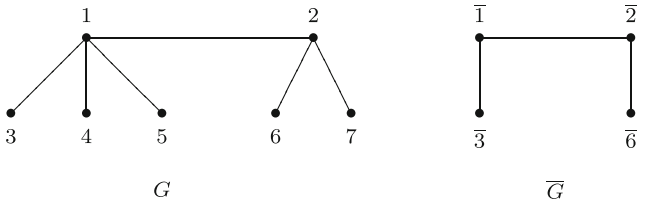


Fig. 4 The graphs G and $\overline{G}$

Proof Let $V(G) = \overline{x_1} \cup \overline{x_2} \cup \cdots \cup \overline{x_m}$ and let h be a lower bound of f and g. Then $h \leq f$ and so by Theorem 2.3, for $x \in V(G)$, $\overline{h(x)} = \overline{f(x)} = \overline{x_j}$ for some j. Thus $h(\overline{x}) \subseteq \overline{h(x)} = \overline{x_j}$ and $f(\overline{x}) \subseteq \overline{f(x)} = \overline{x_j}$. Since $Im(h|_{\overline{x}}) = Im(h) \cap \overline{x_j}$ and $Im(f|_{\overline{x}}) = Im(f) \cap \overline{x_j}$, we have $Im(h|_{\overline{x}}) \subseteq Im(f|_{\overline{x}})$. Similarly we have $Im(h|_{\overline{x}}) \subseteq Im(g|_{\overline{x}})$. Hence, $Im(h|_{\overline{x}}) \subseteq Im(f|_{\overline{x}}) \cap Im(g|_{\overline{x}})$ for all $x \in V(G)$. For the converse, if $Im(f|_{\overline{x}}) \cap Im(g|_{\overline{x}}) \neq \emptyset$ for all $x \in V(G)$, then for each $\overline{x_i}$, there is $y_i \in V(G)$ such that $y_i \in Im(f|_{\overline{x_i}}) \cap Im(g|_{\overline{x_i}})$. We define $h : V(G) \to V(G)$ by $h(x) = y_i$ if $x \in \overline{x_i}$. Then $h \in SEnd(G)$ and $Im(h)$ is a seedling graph. Moreover, $h(x) \in Im(f|_{\overline{x}})$ and $h(x) \in Im(g|_{\overline{x}})$ for all $x \in V(G)$ and so by Lemma 3.2, $h \leq f$ and $h \leq g$. □

Lemma 3.4 *Let $f, g \in X_\mu$ for some $\mu \in Aut(\overline{G})$. If there is $x \in V(G)$ such that $|\overline{x}| = 1$ and $f(x) \neq g(x)$, then f and g have no upper bound.*

Proof Let $f \leq h$ and $g \leq h$. Then $f(x) \in Im(h)$ and so $f(x) = h(y) = f(y)$ for some $y \in V(G)$. Thus $\overline{f(x)} = \overline{f(y)}$ and so $\overline{x} = \overline{y}$. Moreover, $x = y$ for $|\overline{x}| = 1$, and $f(x) = h(x)$. Similarly we have $g(x) = h(x)$, contrary to $f(x) \neq g(x)$. □

Theorem 3.5 *Let $f, g \in X_\mu$ and let $Im(f)$ and $Im(g)$ be seedling graphs. Then for each $\mu \in Aut(\overline{G})$, $x \in V(G)$, $|\overline{x}| = 1$ implies $|\mu(\overline{x})| = 1$ if and only if f and g have an upper bound.*

Proof If $f|_{\overline{x_k}} = g|_{\overline{x_k}}$ for all $\overline{x_k}$, $f = g$. If $f|_{\overline{x_k}} \neq g|_{\overline{x_k}}$ for some $\overline{x_k}$, then $|\mu(\overline{x_k})| > 1$ and so $|\overline{x_k}| > 1$. Let a, b be distinct elements in $\overline{x_k}$ and $f(a) \neq g(a)$. We define $h : V(G) \to V(G)$ by

$$h(x) = \begin{cases} f(a) & \text{if } x = a \\ g(a) & \text{otherwise} \end{cases}.$$

Then $h \in SEnd(G)$ and $h \geq g$ and $h \geq f$. The converse is the special case of Lemma 3.4. □

Example 3.1 We consider a graph G depicted in Fig. 4. We note that $\overline{1} = \{1\}, \overline{2} = \{2\}, \overline{3} = \{3, 4, 5\}$, and $\overline{6} = \{6, 7\}$, and $Aut(\overline{G}) = \{\mu_1, \mu_2\}$ where

$$\mu_1 = \begin{pmatrix} \overline{1} & \overline{2} & \overline{3} & \overline{6} \\ \overline{1} & \overline{2} & \overline{3} & \overline{6} \end{pmatrix} \quad \text{and} \quad \mu_2 = \begin{pmatrix} \overline{1} & \overline{2} & \overline{3} & \overline{6} \\ \overline{2} & \overline{1} & \overline{6} & \overline{3} \end{pmatrix}.$$

Let $f, g \in X_{\mu_1}$, where

$$f = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,3\,3\,3\,6\,6 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,4\,4\,4\,7\,7 \end{pmatrix}.$$

Then the least upper bound is of the form

$$h = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,\epsilon_1\,\epsilon_2\,\epsilon_3\,\delta_1\,\delta_1 \end{pmatrix}$$

where $\epsilon_i = 3, 4$, and $\delta_i = 6, 7$, and not all ϵ_i, δ_i are same. In general, let $f, g \in X_\mu$ and let $Im(f)$ and $Im(g)$ be seedling graphs. Then if $\overline{x_{i_1}}, \overline{x_{i_2}}, \ldots, \overline{x_{i_s}}$ are all equivalence classes of $V(G)$ such that $|\overline{x_{i_j}}| = n_{i_j} > 1$ and $f|_{\overline{x_{i_j}}} \neq g|_{\overline{x_{i_j}}}$, then the number of least upper bounds of f and g is $\Pi_{j=1}^{s}(2^{n_{i_j}} - 2)$.

Lemma 3.6 *Let $f, g \in X_\mu$ for some $\mu \in Aut(\overline{G})$. If there is $c \in V(G)$ such that $|\overline{c}| = 1$ and $f(c) \neq g(c)$, then f and g are not connected.*

Proof We first show that for $h, k \in SEnd(G)$ if $h \leq k$ or $k \leq h$, then $h(c) = k(c)$. Let $h \leq k$ and $h(c) \in Im(h) \subset Im(k)$. Then there is $\alpha \in V(G)$ such that $k(\alpha) = h(c) \in Im(h)$ and so $k(\alpha) = h(\alpha) = h(c)$. Moreover $\overline{\alpha} = \overline{c}$ and so $\alpha = c$ for $|\overline{c}| = 1$, which means $h(c) = k(c)$. We can apply the same argument for the case $k \leq h$. If f and g are connected, there is a finite sequence of elements $\ell_1, \ell_2, \ldots, \ell_m$ in $SEnd(G)$ such that $f = \ell_1, \ell_2, \ldots, \ell_m = g$, and $\ell_i \leq \ell_{i+1}$ or $\ell_{i+1} \leq \ell_i$ for $i = 1, \ldots, m-1$. We note that $f(c) = \ell_2(c) = \cdots = \ell_{m-1}(c) = g(c)$, a contradiction. □

Theorem 3.7 *For a graph G, if $|Aut(\overline{G})| = k$, then $SEnd(G)$ has k connected components if and only if for each $\mu \in Aut(\overline{G})$, $x \in V(G)$, $|\overline{x}| = 1$ implies $|\mu(\overline{x})| = 1$.*

Proof We notice that $SEnd(G)$ has at least k connected components by Lemma 3.1. Suppose that $|\overline{c}| = 1$ and $|\mu(\overline{c})| > 1$ for some $c \in V(G)$ and $\mu \in Aut(\overline{G})$, and $h \in X_\mu$. For two distinct elements a, b in $\mu(\overline{c})$, we define two distinct elements $f, g \in SEnd(G)$ by

$$f(x) = \begin{cases} a & \text{if } x = c \\ h(x) & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} b & \text{if } x = c \\ h(x) & \text{otherwise} \end{cases}.$$

We note that $\overline{f(c)} = \overline{a} = \mu(\overline{c})$, and $\overline{f(x)} = \overline{h(x)} = \mu(\overline{x})$ for $x \neq c$, which means $f \in X_\mu$, and $g \in X_\mu$ by the same argument. Furthermore, f and g are not connected by Lemma 3.6. Hence, X_μ has at least 2 connected components. Therefore, $SEnd(G)$ has more than k connected components.

Conversely, we assume that for each $\mu \in Aut(G_\eta)$, $x \in V(G) = \overline{x_1} \cup \cdots \cup \overline{x_m}$, $|\overline{x}| = 1$ implies $|\mu(\overline{x})| = 1$. We now show that every X_μ is connected. The case of $|X_\mu| = 1$ is clear. Let $|X_\mu| > 1$ and let $f, g \in X_\mu$ be distinct. If $f(x') \neq g(x')$ for some $x' \in \overline{x_k}$, then $\overline{f(x')} = \overline{g(x')} = \mu(\overline{x'})$ for $f, g \in X_\mu$ and $|\mu(\overline{x'})| > 1$,

and so $|\overline{x'}| > 1$ by the assumption. Let a_k, b_k be two distinct elements in $\overline{x_k}$ and $f(a_k) \neq g(a_k)$. We define functions $f_k, g_k : \overline{x_k} \to V(G)$ as follows:

$$f_k(x) = f(a_k) \quad \text{and} \quad g_k(x) = g(a_k).$$

If $f(x') = g(x')$ for all $x' \in \overline{x_k}$, we fix an element $c_k \in Im(f) = Im(g)$. We define functions $f_k, g_k : \overline{x_k} \to V(G)$ as follows:

$$f_k(x) = g_k(x) = c_k.$$

Now we define $f', g' : V(G) \to V(G)$ such that

$$f'|_{\overline{x_k}}(x) = f_k(x) \text{ and } g'|_{\overline{x_k}}(x) = g_k(x).$$

Then $Im(f')$ and $Im(g')$ are seedling graphs, and $f' \leq f$ and $g' \leq g$. By Theorem 3.5, $f' \leq h$ and $g' \leq h$ for some $h \in X_\mu$. Therefore, Thus, f and g are connected by a sequence f, f', h, g', g where

$$f \geq f' \leq h \geq g' \leq g.$$

□

Corollary 3.8 *Let $V(G) = \overline{x_1} \cup \overline{x_2} \cup \cdots \cup \overline{x_m}$ and $|\overline{x_k}| > 1$ for all k. Then $f, g \in X_\mu$ are connected.* □

For example, the complete bipartite graph $K_{m,n}$ $(m, n > 1)$ has two connected components.

Corollary 3.9 *For $\mu \in Aut(\overline{G})$, let $\overline{x_{i_1}}, \overline{x_{i_2}}, \ldots, \overline{x_{i_s}}$ be all equivalence classes of $V(G)$ such that $|\overline{x_{i_j}}| = 1$ and $|\mu(\overline{x_{i_j}})| = k_j > 1$. Then X_μ has $\sum_{j=1}^{s} k_j$ connected components.*

Proof We just note that if $|\overline{x_{i_j}}| = 1$ and $f(x_{i_j}) \neq g(x_{i_j})$ for $f, g \in X_\mu$, then f and g are not comparable. □

Example 3.2 If G is a finite empty graph, then $SEnd(G)$ is connected.

Proof Let $f, g \in SEnd(G)$. If f and g are constant functions defined by $f(x) = a$ and $g(x) = b$ for all $x \in G$, respectively, then a function h on G defined by

$$h(x) = \begin{cases} a & \text{if } x = c \\ b & \text{otherwise} \end{cases}$$

where c is a fixed element in G, is an upper bound of f and g. For a nonconstant function f, we define a constant function ℓ by $\ell(x) = a \in Im(f)$, and then $f \geq \ell$. Hence, any two functions in $SEnd(G)$ are connected. □

References

1. Böttcher, M., Knauer, U.: Endomorphism spectra of graphs. Discr. Math. **109**, 45–57 (1992)
2. Böttcher, M., Knauer, U.: Postscript: endomorphism spectra of graphs. Discr. Math. **270**, 329–331 (2003)
3. Fan, S.: On end-regular graphs. Discr. Math. **159**, 95–102 (1996)
4. Hou, H., Luo, Y.: Graphs whose endomorphism monoids are regular. Discr. Math. **308**, 3888–3896 (2008)
5. Kowol, G., Mitsch, H.: Naturally ordered transformation semigroups. Mh. Math. **102**, 115–138 (1986)
6. Knauer, U.: Endomorphisms of graphs II. Various unretractive graphs. Arch. Math. **55**, 193–203 (1990)
7. Knauer, U.: Unretractive and S-unretractive joins and lexicographic products of graphs. J. Graph Theory **11**(3), 429–440 (1987)
8. Knauer, U., Nieporte, M.: Endomorphisms of graphs I. The monoid of strong endomorphisms. Arch. Math. **52**, 607–614 (1989)
9. Li, W.: A regular endomorphism of a graph and its inverses. Mathematika **41**, 189–198 (1994)
10. Li, W.: Graphs with regular monoids. Discr. Math. **265**, 105–118 (2003)
11. Li, W.: Green's relations on the endomorphism monoid of a graph. Math. Slovaca **45**(4), 335–347 (1995)
12. Li, W.: Green's relations on the strong endomorphism monoid of a graph. Semigroup Forum **47**, 209–214 (1993)
13. Mitsch, H.: A natural partial order for semigroups. Proc. Am. Math. Soc. **97**, 384–388 (1986)
14. Pipattanajinda, N.: The endotype of $(n-3)$-regular graphs of order n. Southeast Asian Bull. Math. **38**, 535–541 (2014)
15. Pipattanajinda, N., Knauer, U., Gyurov, B., Panma, S.: The endomorphism monoids of $(n-3)$-regular graphs of order n. Algebra Discr. Math. **22**(2), 284–300 (2016)
16. Pipattanajinda, N., Knauer, U., Gyurov, B., Panma, S.: The endomorphisms monoids of graphs of order n with a minimum degree $n-3$. Algebra Discr. Math. **18**(2), 274–294 (2014)
17. Subramonian Nambooripad, K.S.: The natural partial order on a regular semigroup. Proc. Edinburgh Math. Soc. (2) **23**(3), 249–260 (1980)
18. Wilkeit, A.: Graphs with regular endomorphism monoid. Arch. Math. **66**, 344–352 (1996)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.