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A novel derivation induced by some binary operations on d-algebras

Thanatporn Bantaojai[1], **Cholatis Suanoom**[2],
Jirayu Phuto[3], **Aiyared Iampan**[4]

[1]Mathematics English Program
Valaya Alongkorn Rajabhat University under the Royal Patronage
Pathumtani 13180, Thailand

[2]Science and Applied Science Center & Program of Mathematics
Kamphaeng Phet Rajabhat University
Kamphaeng Phet 62000, Thailand

[3]Department of Mathematics
Naresuan University
Phitsanulok 65000, Thailand

[4]Department of Mathematics
School of Science
University of Phayao
Phayao 56000, Thailand

email: cholatis.suanoom@gmail.com, jirayup60@email.nu.ac.th,
thanatporn.ban@vru.ac.th, aiyared.ia@up.ac.th

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Abstract

The notion of novel derivations induced by some binary operations of d-algebra is introduced and some important properties are given. The conditions for each regular derivation induced by left or right bi-endomorphisms are obtained. Finally, the composition of derivations are also studied.

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Corresponding author: Aiyared Iampan (aiyared.ia@up.ac.th).
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1 Introduction

In 1999, Neggers and Kim [1] introduced the algebraic structures of d-algebras, a useful generalization of BCK-algebras, and they investigated several relations among d-algebras and BCK-algebras as well as several other relations among d-algebras and oriented digraphs [2]. In 2012, Kandaraj and Chandamouleeswaran [3] introduced the notion of left F-derivations of d-algebras. In 2015 and 2019, Al-Omary et al. [4] introduced the concepts of generalized derivations and (α, β)-derivations on d-algebras.

In this paper, we introduce the notion of novel derivations induced by some binary operations of d-algebra, give some important properties, and find some conditions for each regular derivation induced by left or right bi-endomorphisms. Finally, we study the composition of two derivations.

2 Preliminaries

In this section, we will review the definitions, theorems and the knowledge needed for our study.

Definition 2.1. *[1] An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called a d-algebra, where X is a nonempty set, $*$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*
(d1) $(\forall x \in X)(x * x = 0)$,
(d2) $(\forall x \in X)(0 * x = 0)$,
(d3) $(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)$.
On a d-algebra $X = (X, *, 0)$, the binary relation $\leq$ is defined as follows:
$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0)$.

Example 2.2. *Let $X = \{0, a, b, c\}$ with the following Cayley table:*

$*$	0	a	b	c
0	0	0	0	0
a	c	0	a	a
b	b	c	0	0
c	c	b	a	0

Then $X = (X, *, 0)$ is a d-algebra.

Definition 2.3. *[2] A nonempty subset S of a d-algebra $X = (X, *, 0)$ is called a subalgebra of X if $(\forall x, y \in S)(x * y \in S)$, and a d-ideal of X if*
(I1) $(\forall x, y \in X)(x * y \in S, y \in S \Rightarrow x \in S)$,

(I2) $(\forall x \in S, \forall y \in X)(x * y \in S)$.
It is easy to check that $\{0\}$ *and* X *are d-ideals of* X.

We know that if I is a d-ideal of a d-algebra $X = (X, *, 0)$, then $0 \in I$, and every d-ideal of X is a subalgebra.

Definition 2.4. *[1] A d-algebra* $X = (X, *, 0)$ *is said to be an edge if* $(\forall x \in X)(x * X = \{x, 0\})$. *It is known that if* X *is an edge d-algebra, then* $x * 0 = x$ *for all* $x \in X$.

Proposition 2.5. *[3] In any d-algebra* $X = (X, *, 0)$, *the following properties hold:*
(d4) $(\forall x, y, z \in X)((x * y) * z = (x * z) * y)$,
(d5) $(\forall x, y \in X)(x * (x * y) \leq y)$,
(d6) $(\forall x, y, z \in X)((x * y) * z \leq x * (y * z))$,
(d7) $(\forall x, y, z \in X)((x * y) * (x * z) \leq z * y)$,
(d8) $(\forall x, y, z \in X)((x * z) * (y * z) \leq x * y)$,
(d9) $(\forall x \in X)(x \leq 0 \Rightarrow x = 0)$,
(d10) $(\forall x, y, z \in X)(x * y = x * z \Rightarrow y = z)$,
(d11) $(\forall x, y, z \in X)(y * x = z * x \Rightarrow y = z)$,
(d12) $(\forall x, y, z \in X)(x \leq y \Rightarrow x * z \leq y * z \text{ and } z * y \leq z * x)$.

Definition 2.6. *[6] A d-algebra* $X = (X, *, 0)$ *is said to be associative if* $(\forall x, y, z \in X)((x * y) * z = x * (y * z))$, *and edge if* $(\forall x \in X)(x * X = \{x, 0\})$. *It is known that if* $X = (X, *, 0)$ *is an edge d-algebra, then (E)* $(\forall x \in X)(x * 0 = x)$.

3 Main results

In this section, we introduce bi-endomorphisms on d-algebras. After this, we will use X instead of a d-algebra $(X, *, 0)$.

Definition 3.1. *A binary operation* f *on* X *is called a left bi-endomorphism on* X *if* $(\forall x, y, z \in X)(f(x*y, z) = f(x, z)*f(y, z))$, *a right bi-endomorphism on* X *if* $(\forall x, y, z \in X)(f(x, y*z) = f(x, y)*f(x, z))$, *and a bi-endomorphism on* X *if it is both a left and a right bi-endomorphism on* X.

Example 3.2. *In Example 2.2, we define a binary operation* f *on* X *by*

$$f(x, y) = \begin{cases} c & \text{if } (x, y) = (c, 0), \\ b & \text{if } (x, y) = (b, 0), \\ a & \text{if } (x, y) = (a, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Then f is a left bi-endomorphism on X.

Proposition 3.3. *If f is a left bi-endomorphism on X, then*

(1) $(\forall x \in X)(f(0, x) = 0)$,

(2) $(\forall x, y \in X)(0 \leq f(x, y))$,

(3) $(\forall x, y, z \in X)(x \leq y \Rightarrow f(x, z) \leq f(y, z))$.

Proof. (1) Let $x \in X$. By (d1), we have $f(0, x) = f(x * x, x) = f(x, x) * f(x, x) = 0$.

(2) It follows from (d2).

(3) Let $x, y, z \in X$ be such that $x \leq y$. By 3.3, we have $0 = f(0, z) = f(x * y, z) = f(x, z) * f(y, z)$. Hence, $f(x, z) \leq f(y, z)$. □

For right bi-endomorphisms, it can be deduced from Proposition 3.3.

Proposition 3.4. *If f is a right bi-endomorphism on X, then*

(1) $(\forall x \in X)(f(x, 0) = 0)$,

(2) $(\forall x, y \in X)(0 \leq f(x, y))$,

(3) $(\forall x, y, z \in X)(x \leq y \Rightarrow f(z, x) \leq f(z, y))$.

If β_l is a left bi-endomorphism on X, we define $\ker(\beta_l) = \{x \in X : \beta_l(x, 0) = 0\}$. If β_l is a right bi-endomorphism on X, we define $\ker(\beta_r) = \{x \in X : \beta_r(0, x) = 0\}$. If β is a bi-endomorphism on X, we define $\ker(\beta) = \{x \in X : \beta(x, x) = 0\}$. By Propositions 3.3 and 3.4, we have $\ker(\beta_l)$, $\ker(\beta_r)$, and $\ker(\beta)$ are nonempty.

Proposition 3.5. (1) *If β_l is a left bi-endomorphism on X, then* $\ker(\beta_l)$ *is a d-ideal (and also a subalgebra) of X.*

(2) *If β_r is a right bi-endomorphism on X, then* $\ker(\beta_r)$ *is a d-ideal (and also a subalgebra) of X.*

(3) *If β is a bi-endomorphism on X, then* $\ker(\beta)$ *is a subalgebra of X.*

Proof. (1) (I1) Suppose that β_l is a left bi-endomorphism on X. Let $x * y \in \ker(\beta_l)$ and $y \in \ker(\beta_l)$. Then $0 = \beta_l(x*y, 0) = \beta_l(x, 0)*\beta_l(y, 0) = \beta_l(x, 0)*0$. Using (d9), we get $\beta_l(x, 0) = 0$. Thus $x \in \ker(\beta_l)$.
(I2) Let $x \in \ker(\beta_l)$ and $y \in X$. By (d2), we have $\beta_l(x * y, 0) = \beta_l(x, 0) *$

$\beta_l(y,0) = 0 * \beta_l(y,0) = 0$. Thus $x * y \in \ker(\beta_l)$. Therefore, $\ker(\beta_l)$ is a d-ideal of X.

(2) By the same method of 3.5, the statement 3.5 can be obtained.

(3) Suppose that β is a bi-endomorphism on X. Let $x, y \in \ker(\beta)$. By (d2), we have $\beta(x * y, x * y) = \beta(x, x * y) * \beta(y, x * y) = (\beta(x,x) * \beta(x,y)) * (\beta(y,x) * \beta(y,y)) = (0 * \beta(x,y)) * (\beta(y,x) * 0) = 0 * (\beta(y,x) * 0) = 0$. Thus $x * y \in \ker(\beta)$. Therefore, $\ker(\beta)$ is a subalgebra of X. □

Let $B_l(X)$ (resp., $B_r(X)$, $B(X)$) be a set of all left bi-endomorphisms (resp., right bi-endomorphisms, bi-endomorphisms) on an associative d-algebra X. We define an operation $\star$ on $B_l(X)$ by for all $f, g \in B_l(X)$, $(\forall x, y \in X)((f \star g)(x,y) = f(x,y) * g(x,y))$. Indeed, let $x, y, z \in X$ and $f, g \in B_l(X)$. By (d4) and associativity, we have $(f \star g)(x * y, z) = f(x * y, z) * g(x * y, z) = (f(x,z) * f(y,z)) * (g(x,z) * g(y,z)) = (f(x,z) * (g(x,z) * g(y,z))) * f(y,z) = ((f(x,z) * g(x,z)) * g(y,z)) * f(y,z) = ((f \star g)(x,z) * g(y,z)) * f(y,z) = ((f \star g)(x,z) * f(y,z)) * g(y,z) = (f \star g)(x,z) * (f(y,z) * g(y,z)) = (f \star g)(x,z) * (f \star g)(y,z)$. Thus $f \star g \in B_l(X)$, so $\star$ is a binary operation on $B_l(X)$. Similarly, we define an operation $\star$ on $B_r(X)$ and on $B(X)$ as an operation $\star$ on $B_l(X)$.

Theorem 3.6. *If X is an associative d-algebra, then $(B_l(X), \star, O)$, $(B_r(X), \star, O)$, and $(B(X), \star, O)$ are also associative d-algebras, where O is the zero map; i.e., $O(x,y) = 0$ for all $x, y \in X$.*

Proof. (d1) Let $f \in B_l(X)$ and $x, y \in X$. By (d1), we have $(f \star f)(x,y) = f(x,y) * f(x,y) = 0$. Thus $f \star f = O$.
(d2) Let $f \in B_l(X)$ and $x, y \in X$. By (d2), we have $(O \star f)(x,y) = O(x,y) * f(x,y) = 0 * f(x,y) = 0$. Thus $O \star f = O$.
(d3) Let $f, g \in B_l(X)$ be such that $f \star g = O$ and $g \star f = O$ and $x, y \in X$. Then $f(x,y) * g(x,y) = (f \star g)(x,y) = 0 = (g \star f)(x,y) = g(x,y) * f(x,y)$. By (d3), we have $f(x,y) = g(x,y)$. Thus $f = g$. (Associativity) Let $f, g, h \in B_l(X)$ and $x, y \in X$. Then $((f \star g) \star h)(x,y) = (f(x,y) * g(x,y)) * h(x,y) = f(x,y) * (g(x,y) * h(x,y)) = (f \star (g \star h))(x,y)$. Thus, $(f \star g) \star h = f \star (g \star h)$. Hence, $(B_l(X), \star, O)$ is an associative d-algebra. For $B_r(X)$ and $B(X)$, it can be proved the same as $B_l(X)$. □

A self-map $\phi : X \to X$ is called an *endomorphism* on X if $\phi(x * y) = \phi(x) * \phi(y)$, for all $x, y \in X$.

Definition 3.7. *Let $\phi : X \to X$ be an endomorphism on X. A binary operation f on X is called a ϕ-endomorphism on X if $(\forall x, y \in X)(f(x,y) = \phi(x) * \phi(y))$.*

Proposition 3.8. *Let $\phi : X \to X$ be an endomorphism on X.*

(1) *If a left bi-endomorphism β_l on X is a ϕ-endomorphism such that $\beta_l(x, 0) = 0$ for all $x \in X$, then ϕ is the zero map.*

(2) *If a right bi-endomorphism β_r on X is a ϕ-endomorphism, then $\beta_r(0, x) = 0$ for all $x \in X$ and ϕ is the zero map.*

Proof. (1) Suppose that a left bi-endomorphism β_l on X is a ϕ-endomorphism such that $\beta_l(x, 0) = 0$ for all $x \in X$. Let $x \in X$. Then $0 = \beta_l(x, 0) = \phi(x) * \phi(0) = \phi(x) * 0$. As (d9), we get $\phi(x) = 0$. Hence, ϕ is the zero map.

(2) Suppose that a right bi-endomorphism β_r on X is a ϕ-endomorphism. Let $x \in X$. By (d2), we have $\beta_r(0, x) = \phi(0) * \phi(x) = 0 * \phi(x) = 0$. By Proposition 3.4 3.4, we have $0 = \beta_r(x, 0) = \phi(x) * \phi(0) = \phi(x) * 0$. Using (d9), we have $\phi(x) = 0$. Hence, ϕ is the zero map. □

The binary operation $\sqcap$ on X is defined by $(\forall x, y \in X)(x \sqcap y = (y * x) * x)$.

Next, we generalize derivation on d-algebra induced by a binary operation f on X.

Definition 3.9. *Consider a binary operation f on X. A self-map $d_f : X \to X$ is called an (l, r)-f-derivation of X if $(\forall x, y \in X)(d_f(x * y) = (d_f(x) * f(x, y)) \sqcap (f(x, y) * d_f(y)))$, an (r, l)-f-derivation of X if $(\forall x, y \in X)(d_f(x * y) = (f(x, y) * d_f(y)) \sqcap (d_f(x) * f(x, y)))$, an f-derivation of X if it is both an (l, r)- and an (r, l)-f-derivation of X, and regular if $d_f(0) = 0$.*

Theorem 3.10. *If d_f is a regular (l, r)-f-derivation of X, then $(\forall x \in X)(f(0, x) \leq d_f(x))$.*

Proof. Suppose that d_f is a regular (l, r)-f-derivation of X. Let $x \in X$. By (d2), we have $0 = d_f(0) = d_f(0 * x) = (d_f(0) * f(0, x)) \sqcap (f(0, x) * d_f(x)) = (0 * f(0, x)) \sqcap (f(0, x) * d_f(x)) = 0 \sqcap (f(0, x) * d_f(x)) = ((f(0, x) * d_f(x)) * 0) * 0$. By (d9), we have $f(0, x) * d_f(x) = 0$, that is, $f(0, x) \leq d_f(x)$. □

Theorem 3.11. *If d_f is a regular (r, l)-f-derivation of X, then $(\forall x \in X)((d_f(x) * f(x, x)) * (f(x, x) * d_f(x)) \leq (f(x, x) * d_f(x)))$.*

Proof. Suppose that d_f is regular. Let $x \in X$. By (d1), we have $0 = d_f(0) = d_f(x * x) = (f(x, x) * d_f(x)) \sqcap (d_f(x) * f(x, x)) = ((d_f(x) * f(x, x)) * (f(x, x) * d_f(x))) * (f(x, x) * d_f(x))$. That is, $(d_f(x) * f(x, x)) * (f(x, x) * d_f(x)) \leq (f(x, x) * d_f(x))$. □

Proposition 3.12. *Let β_l be a left bi-endomorphism on X. Every (l,r)-β_l-derivation of X is regular. In particular, if X is edge, then $(\forall x \in X)(d_{\beta_l}(x) = (d_{\beta_l}(x) * \beta_l(x,0)) \sqcap \beta_l(x,0))$.*

Proof. Suppose that d_{β_l} is an (l,r)-β_l-derivation of X. By (d1), (d2), and Propositions 3.3 3.3, we have $d_{\beta_l}(0) = d_{\beta_l}(0*0) = (d_{\beta_l}(0)*\beta_l(0,0)) \sqcap (\beta_l(0,0)* d_{\beta_l}(0)) = (d_{\beta_l}(0) * 0) \sqcap (0 * d_{\beta_l}(0)) = (d_{\beta_l}(0) * 0) \sqcap 0 = (0 * (d_{\beta_l}(0) * 0)) * (d_{\beta_l}(0) * 0) = 0$. Hence, d_{β_l} is regular.

Suppose that X is edge. Let $x \in X$. By (E), we have $d_{\beta_l}(x) = d_{\beta_l}(x*0) = (d_{\beta_l}(x) * \beta_l(x,0)) \sqcap (\beta_l(x,0) * d_{\beta_l}(0)) = (d_{\beta_l}(x) * \beta_l(x,0)) \sqcap (\beta_l(x,0) * 0) = (d_{\beta_l}(x) * \beta_l(x,0)) \sqcap \beta_l(x,0)$. □

Definition 3.13. *Let I be a d-ideal of X, g a self-map on X, and f a binary operation on X. Then I is called an f-ideal of X if $f(x,y) \in I$ for all $x,y \in I$. An f-ideal I of X is called to be g-invariant if $g(I) \subseteq I$.*

Corollary 3.14. *Let β_l be a left bi-endomorphism on an associative and edge d-algebra X. Every β_l-ideal of X is d_{β_l}-invariant.*

Proof. Let I be a β_l-ideal of X. Let $x \in I$. By (d1), (E), associativity, and Proposition 3.12, we have $d_{\beta_l}(x) = (d_{\beta_l}(x) * \beta_l(x,0)) \sqcap \beta_l(x,0) = (\beta_l(x,0) * (d_{\beta_l}(x)*\beta_l(x,0))) * (d_{\beta_l}(x)*\beta_l(x,0)) = \beta_l(x,0)*((d_{\beta_l}(x)*\beta_l(x,0))*(d_{\beta_l}(x)* \beta_l(x,0))) = \beta_l(x,0) * 0 = \beta_l(x,0) \in I$. Hence, I is d_{β_l}-invariant. □

Proposition 3.15. *Let β_r be a right bi-endomorphism on an edge d-algebra X. Every (l,r)-β_r-derivation of X is the zero map.*

Proof. Suppose that d_{β_r} is an (l,r)-β_r-derivation of X. Let $x \in X$. By (E), (d2), and Propositions 3.4 3.4, we have $d_{\beta_r}(x) = d_{\beta_r}(x * 0) = (d_{\beta_r}(x) * \beta_r(x,0)) \sqcap (\beta_r(x,0) * d_{\beta_r}(0)) = (d_{\beta_r}(x) * 0) \sqcap (0 * d_{\beta_r}(0)) = d_{\beta_r}(x) \sqcap 0 = (0 * d_{\beta_r}(0)) * d_{\beta_r}(0) = 0$. Hence, d_{β_r} is the zero map. □

Theorem 3.16. *Let β_l be a left bi-endomorphism on an edge d-algebra X. If d_{β_l} is an (r,l)-β_l-derivation of X satisfying $d_{\beta_l}(x) = \beta_l(x,0)$ for all $x \in X$, then d_{β_l} is the zero map.*

Proof. Suppose that d_{β_l} is an (r,l)-β_l-derivation of X satisfying $d_{\beta_l}(x) = \beta_l(x,0)$ for all $x \in X$. Let $x \in X$. By (E), (d1), and (d2), we have $d_{\beta_l}(x) = d_{\beta_l}(x*0) = (\beta_l(x,0)*d_{\beta_l}(0)) \sqcap (d_{\beta_l}(x)*\beta_l(x,0)) = (\beta_l(x,0)*d_{\beta_l}(0)) \sqcap (d_{\beta_l}(x)* d_{\beta_l}(x)) = (\beta_l(x,0)*d_{\beta_l}(0)) \sqcap 0 = (0*(\beta_l(x,0)*d_{\beta_l}(0)))*(\beta_l(x,0)*d_{\beta_l}(0)) = 0$. Hence, d_{β_l} is the zero map. □

Theorem 3.17. *Let β_l be a left bi-endomorphism on an edge d-algebra X. If d_{β_l} is an (l,r)-β_l-derivation of X satisfying $d_{\beta_l}(0) = \beta_l(x,0)$ for all $x \in X$, then d_{β_l} is the zero map.*

Proof. Suppose that d_{β_l} is an (l,r)-β_l-derivation of X satisfying $d_{\beta_l}(0) = \beta_l(x,0)$ for all $x \in X$. Let $x \in X$. By (E), (d1), and (d2), we have $d_{\beta_l}(x) = d_{\beta_l}(x*0) = (d_{\beta_l}(x)*\beta_l(x,0)) \sqcap (\beta_l(x,0)*d_{\beta_l}(0)) = (d_{\beta_l}(x)*\beta_l(x,0)) \sqcap (d_{\beta_l}(0)*d_{\beta_l}(0)) = (d_{\beta_l}(x)*\beta_l(x,0)) \sqcap 0 = (0*(d_{\beta_l}(x)*\beta_l(x,0)))*(d_{\beta_l}(x)*\beta_l(x,0)) = 0$. Hence, d_{β_l} is the zero map. □

Theorem 3.18. *Let β_r be a right bi-endomorphism on an edge d-algebra X. If d_{β_r} is an (r,l)-β_r-derivation of X satisfying $d_{\beta_r}(x) = \beta_r(0,x)$ for all $x \in X$, then d_{β_r} is the zero map.*

Proof. Suppose that d_{β_r} is an (r,l)-β_r-derivation of X satisfying $d_{\beta_r}(x) = \beta_r(0,x)$ for all $x \in X$. Let $x \in X$. By (E), (d1), and (d2), we have $d_{\beta_r}(0)*0 = d_{\beta_r}(0) = d_{\beta_r}(0*x) = (\beta_r(0,x)*d_{\beta_r}(x)) \sqcap (d_{\beta_r}(0)*\beta_r(0,x)) = (d_{\beta_r}(x)*d_{\beta_r}(x)) \sqcap (d_{\beta_r}(0)*d_{\beta_r}(x)) = 0 \sqcap (d_{\beta_r}(0)*d_{\beta_r}(x)) = ((d_{\beta_r}(0)*d_{\beta_r}(x))*0)*0 = d_{\beta_r}(0)*d_{\beta_r}(x)$. Using (d10), we have $d_{\beta_r}(x) = 0$. Hence, d_{β_r} is the zero map. □

Theorem 3.19. *Let β_r be a right bi-endomorphism on an edge d-algebra X. If d_{β_r} is an (l,r)-β_r-derivation of X, then d_{β_r} is the zero map.*

Proof. Suppose that d_{β_r} is an (l,r)-β_r-derivation of X. Let $x \in X$. By (E), (d1), (d2), and Propositions 3.4 3.4, we have $d_{\beta_r}(x) = d_{\beta_r}(x*0) = (d_{\beta_r}(x)*\beta_r(x,0)) \sqcap (\beta_r(x,0)*d_{\beta_r}(0)) = (d_{\beta_r}(x)*0) \sqcap (0*d_{\beta_r}(0)) = d_{\beta_r}(x) \sqcap 0 = (0*d_{\beta_r}(x))*d_{\beta_r}(x) = 0$. Hence, d_{β_r} is the zero map. □

Next, we determine the composition of derivations on d-algebras.

Proposition 3.20. *Let f be a binary operation on an associative and edge d-algebra X defined by $f(x,y) = x$ for all $x, y \in X$. If d_f and D_f are (l,r)-f-derivations of X, then $d_f \circ D_f$ is also an (l,r)-f-derivation of X.*

Proof. Suppose that d_f and D_f are (l,r)-f-derivations of X. Let $x, y \in X$. By associativity, (d1), and (E), we have $(d_f \circ D_f)(x*y) = d_f(D_f(x*y)) = d_f((D_f(x)*f(x,y)) \sqcap (f(x,y)*D_f(y))) = d_f((D_f(x)*x) \sqcap (x*D_f(y))) = d_f(((x*D_f(y))*(D_f(x)*x))*(D_f(x)*x)) = d_f((x*D_f(y))*((D_f(x)*x)*(D_f(x)*x))) = d_f((x*D_f(y))*0) = d_f(x*D_f(y)) = (d_f(x)*f(x,D_f(y))) \sqcap (f(x,D_f(y))*d_f(D_f(y))) = (d_f(x)*f(x,D_f(y))) \sqcap (f(x,D_f(y))*(d_f \circ D_f)(y)) = (d_f(x)*f(x,y)) \sqcap (f(x,y)*(d_f \circ D_f)(y)) =$

$((f(x,y)*(d_f\circ D_f)(y))*(d_f(x)*f(x,y)))*(d_f(x)*f(x,y)) = (f(x,y)*(d_f\circ D_f)(y))*((d_f(x)*f(x,y))*(d_f(x)*f(x,y))) = (f(x,y)*(d_f\circ D_f)(y))*0 = f(x,y)*(d_f\circ D_f)(y) = f(x,D_f(y))*(d_f\circ D_f)(y) = f(x,y)*(d_f\circ D_f)(y) = (f(x,y)*(d_f\circ D_f)(y))*0 = (f(x,y)*(d_f\circ D_f)(y))*(((d_f\circ D_f)(x)*f(x,y))*((d_f\circ D_f)(x)*f(x,y))) = ((f(x,y)*(d_f\circ D_f)(y))*((d_f\circ D_f)(x)*f(x,y)))*((d_f\circ D_f)(x)*f(x,y)) = ((d_f\circ D_f)(x)*f(x,y))\sqcap(f(x,y)*(d_f\circ D_f)(y))$. Hence, $d_f\circ D_f$ is an (l,r)-f-derivation of X. □

The following proposition follows easily.

Proposition 3.21. *Let f be a binary operation on defined by $f(x,y) = y$ for all $x,y\in X$. If d_f and D_f are (r,l)-f-derivations of X, then $d_f\circ D_f$ is also an (r,l)-f-derivation of X.*

Theorem 3.22. *Let β_l be a left bi-endomorphism on an associative and edge d-algebra X and let d_{β_l}, D_{β_l} be (l,r)-β_l-derivations of X. Then*

(1) $(\forall x\in X)((d_{\beta_l}\circ D_{\beta_l})(x) = d_{\beta_l}(\beta_l(x,0)))$,

(2) $(\forall x\in X)(\beta_l(x,0) = 0 \Rightarrow (d_{\beta_l}\circ D_{\beta_l})(x) = 0)$,

(3) $d_{\beta_l}\circ D_{\beta_l}$ is regular.

Proof. (1) Let $x\in X$. By (E), (d1), and Proposition 3.12, we have $(d_{\beta_l}\circ D_{\beta_l})(x) = (d_{\beta_l}\circ D_{\beta_l})(x*0) = d_{\beta_l}(D_{\beta_l}(x*0)) = d_{\beta_l}((D_{\beta_l}(x)*\beta_l(x,0))\sqcap(\beta_l(x,0)*D_{\beta_l}(0))) = d_{\beta_l}(((\beta_l(x,0)*D_{\beta_l}(0))*(D_{\beta_l}(x)*\beta_l(x,0)))*(D_{\beta_l}(x)*\beta_l(x,0))) = d_{\beta_l}((\beta_l(x,0)*D_{\beta_l}(0))*((D_{\beta_l}(x)*\beta_l(x,0))*(D_{\beta_l}(x)*\beta_l(x,0)))) = d_{\beta_l}((\beta_l(x,0)*D_{\beta_l}(0))*0) = d_{\beta_l}(\beta_l(x,0)*D_{\beta_l}(0)) = d_{\beta_l}(\beta_l(x,0)*0) = d_{\beta_l}(\beta_l(x,0))$.

(2) The proof is straightforward by (1) and Proposition 3.12.

(3) This follows from (2) and Propositions 3.3 3.3. □

From Proposition 3.15, we get the following theorem.

Theorem 3.23. *Let β_r be a right bi-endomorphism on an edge d-algebra X. If d_{β_r} and D_{β_r} are (l,r)-β_r-derivations of X, then $d_{\beta_l}\circ D_{\beta_l}$ is the zero map.*

Next, we define $\mathrm{Der}_{(l,r)\text{-}f}(X)$ (resp., $\mathrm{Der}_{(r,l)\text{-}f}(X)$) denotes the set of all (l,r)-f-derivations (resp., (r,l)-f-derivations) on an associative and edge d-algebra X. Let $d_f, D_f\in\mathrm{Der}_{(l,r)\text{-}f}(X)$. Define the binary operation $\Cap$ on $\mathrm{Der}_f(X)$ as follows: $d_f\Cap D_f = D_f$.

The following two propositions follow from Propositions 3.12 and 3.15, respectively.

Proposition 3.24. *Let β_l be a left bi-endomorphism on an associative and edge d-algebra X. If d_{β_l} and D_{β_l} are (l,r)-β_l-derivations of X, then $d_{\beta_l} \Cap D_{\beta_l}$ is a regular (l,r)-β_l-derivation on X. Moreover, $(\mathrm{Der}_{(l,r)\text{-}\beta_l}(X), \Cap)$ is a band (idempotent semigroup).*

Proposition 3.25. *Let β_r be a right bi-endomorphism on an associative and edge d-algebra X. If d_{β_r} and D_{β_r} are (l,r)-β_r-derivations of X, then $d_{\beta_r} \Cap D_{\beta_r}$ is the zero map. Moreover, $(\mathrm{Der}_{(l,r)\text{-}\beta_r}(X), \Cap)$ is a zero semigroup.*

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