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On Bd-algebras

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Abstract

In this paper, we introduce the new algebra structure which combines properties from B-algebras and d-algebras: An algebra $\mathcal{X} = (\mathcal{X}, \circ, 0)$ is called a *Bd-algebra* if it satisfies the following axioms for all $x, y \in \mathcal{X}$: $x \circ 0 = x$, and if $x \circ y = 0$ and $y \circ x = 0$, then $x = y$. In addition, we give some properties of Bd-ideals and Bd-subalgebras of Bd-algebras and construction of quotient algebras.

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1 Introduction and Preliminaries

In 1966, Imai and Iséki [9, 8] defined BCK and, more generally, BCI-algebras). These algebras have been extensively studied. In 1991, Neggers and Kim [4] generalized BCK-algebras in another direction as d-algebras. In 2002, Neggers and Kim [7] gave properties of B-algebras. Such algebra took some properties from BCI and BCH-algebras (see [2]). In 2005, Kim and Park [6] showed that the class of 0-commutative B-algebras is the class of semisimple BCI-algebras. From two algebras, we have the same property; i.e., $x \circ x = 0$, where x is an element in such algebra. Moreover, there are many algebras that some properties are from B-algebra and d-algebras. In 2017, Jun et al. [5] introduced a new algebras from a B-algebra; namely, BH-algebra. Later, Saeid et al. [1] gave a new algebra; i.e., BI-algebra. Moreover, they compared such algebra with other algebras; i.e., $BCI/BCK/BCH/BH/BZ/$d$/Q/B/BM/BO/BG/BP/BN/BF$-algebras.

In 2002, Neggers and Kim [7] introduced B-algebra and combined some properties from BCI and BCH-algebras for B-algebra as follows:

Definition 1.1. *[7] A B-algebra is an algebra $\mathcal{X} = (\mathcal{X}, \circ, 0)$ of type $(2, 0)$ satisfying the following: (B1) $(\forall x \in \mathcal{X})(x \circ x = 0)$, (B2) $(\forall x \in \mathcal{X})(x \circ 0 = x)$, and (B3) $(\forall x, y, z \in \mathcal{X})((x \circ y) \circ z = x \circ (z \circ (0 \circ y)))$.*

In 1991, Neggers and Kim generalized BCK-algebras and gave a new algebra; i.e., d-algebra:

Definition 1.2. *[4] A d-algebra is an algebra $\mathcal{X} = (\mathcal{X}, \circ, 0)$ of type $(2, 0)$ satisfying the following:*
(B1) $(\forall x \in \mathcal{X})(x \circ x = 0)$, (d1) $(\forall x \in \mathcal{X})(0 \circ x = 0)$, and (d2) $(\forall x, y \in \mathcal{X})(x \circ y = 0, y \circ x = 0 \Rightarrow x = y)$.

In this paper, we introduce the new algebra structure which combines properties from B-algebras and d-algebras: An algebra $\mathcal{X} = (\mathcal{X}, \circ, 0)$ is called a *Bd-algebra* if it satisfies the following axioms for all $x, y \in \mathcal{X}$: $x \circ 0 = x$, if $x \circ y = 0$ and $y \circ x = 0$, then $x = y$. In addition, we give some properties of Bd-ideals and Bd-subalgebras of Bd-algebras and construction of quotient algebras.

2 Main results

We construct a new algebra which combines from B-algebras (B2) and d-algebras (d2) as follows:

Definition 2.1. *A Bd-algebra is an algebra* $\mathcal{X} = (\mathcal{X}, \circ, 0)$ *of type* $(2, 0)$ *satisfying the following: (Bd1)* $(\forall x \in \mathcal{X})(x \circ 0 = x)$ *and (Bd2)* $(\forall x, y \in \mathcal{X})(x \circ y = 0$ *and* $y \circ x = 0 \Rightarrow x = y)$.

These axioms played important roles for researchers to construct algebraic structures and investigate several properties [5, 3, 1].

Definition 2.2. *An algebra* $\mathcal{X} = (\mathcal{X}, \circ, 0)$ *of type* $(2, 0)$ *is called a*

(1) *BH-algebra if it satisfies (Bd1), (Bd2), and (B1),*

(2) *BCI-algebra if it satisfies (Bd1), (Bd2), and* $(\forall x, y, z \in \mathcal{X})(((x \circ y) \circ (x \circ z)) \circ (z \circ y) = 0)$,

(3) *BZ-algebra if it satisfies (Bd1), (Bd2), and* $(\forall x, y, z \in \mathcal{X})(((x \circ z) \circ (y \circ z)) \circ (x \circ y) = 0)$.

Remark 2.3. *Every BH/BCI/BZ-algebra is a Bd-algebra.*

Example 2.4. *Let* $\mathcal{X} = \{0, a, b, c\}$ *be a set with a binary operation* $\circ$ *defined by the following Cayley table:*

$\circ$	0	a	b	c
0	0	0	c	0
a	a	0	b	c
b	b	b	b	c
c	c	b	b	c

Then $\mathcal{X} = (\mathcal{X}, \circ, 0)$ *is a Bd-algebra. However,* $\mathcal{X}$ *is not a BH-algebra because* $b \circ b = b \neq 0$. *As* $((b \circ b) \circ (b \circ b)) \circ (b \circ b) = b \neq 0$, *it is not a BCI/BZ-algebra.*

We'll refer to $\mathcal{X}$ as a *Bd*-algebra $(\mathcal{X}, \circ, 0)$ from now on.

Definition 2.5. $\mathcal{X}$ *is said to be*

(1) *edge if* $x \circ \mathcal{X} = \{0, x\}$ *for all* $x \in \mathcal{X}$, *where* $x \circ \mathcal{X} = \{x \circ a : a \in \mathcal{X}\}$,

(2) *commutative if* $(\forall x, y \in \mathcal{X})(x \circ y = y \circ x)$,

(3) 0*-commutative if* $(\forall x, y \in \mathcal{X})(x \circ (0 \circ y) = y \circ (0 \circ x))$,

(4) *associative if* $(\forall x, y \in \mathcal{X})((x \circ y) \circ z = x \circ (y \circ z))$,

(5) *medial if* $(\forall x, y, z, u \in \mathcal{X})((x \circ y) \circ (z \circ u) = (x \circ z) \circ (y \circ u))$.

The following proposition is direct from the definition of Bd-algebras.

Proposition 2.6. *The following properties hold in* $\mathcal{X}$.

(Bd3) $0 \circ 0 = 0$,

(Bd4) $(\forall x \in \mathcal{X})(x \circ 0 = 0 \Rightarrow \mathcal{X} = \{0\})$,

(Bd5) $(\forall x \in \mathcal{X})(x \circ (0 \circ x) = 0, (0 \circ x) \circ x = 0 \Rightarrow 0 \circ x = x = x \circ 0)$,

(Bd6) $(\forall x, y \in \mathcal{X})((x \circ y) \circ (y \circ x) = 0, (y \circ x) \circ (x \circ y) = 0) \Leftrightarrow \mathcal{X}$ *is commutative,*

(Bd7) $(\forall x, y \in \mathcal{X})(\mathcal{X}$ *is commutative and* $x \circ y = 0 \Rightarrow x = y)$.

The following proposition follows directly from (Bd1).

Proposition 2.7. *If* $\mathcal{X}$ *is commutative, then*

(1) $(\forall x, y \in \mathcal{X})(x \circ y = (0 \circ y) \circ (0 \circ x))$,

(2) $(\forall x \in \mathcal{X})(x = 0 \circ (0 \circ x))$,

(3) $\mathcal{X}$ *is* 0*-commutative.*

The following theorem obtains from Proposition 2.7 and using (Bd1).

Theorem 2.8. *Assume* $\mathcal{X}$ *is associative. Then* $\mathcal{X}$ *is commutative if and only if* $\mathcal{X}$ *is* 0*-commutative.*

The following proposition follows directly from (Bd1).

Proposition 2.9. *If* $\mathcal{X}$ *is medial, then*

(1) $(\forall x, y \in \mathcal{X})(0 \circ (x \circ y) = (0 \circ x) \circ (0 \circ y))$,

(2) $(\forall x, y, z \in \mathcal{X})((x \circ y) \circ z = (x \circ z) \circ y)$,

(3) $(\forall x, y, z \in \mathcal{X})(x \circ (y \circ z) = (x \circ y) \circ (0 \circ z))$,

(4) *if* $\mathcal{X}$ *is associative satisfying (B1), then* $\mathcal{X}$ *is a* B*-algebra.*

Now, we determine properties of Bd-subalgebras and Bd-ideals of a Bd-algebra and construct quotient Bd-algebras with some properties.

Definition 2.10. *A non-empty subset* S *of* $\mathcal{X}$ *is called*

(1) *a Bd-subalgebra of* $\mathcal{X}$ *if (S1)* $0 \in S$ *and (S2)* $(\forall x, y \in S)(x \circ y \in S)$,

(2) *a Bd-ideal of* $\mathcal{X}$ *if it satisfies (S1), (I1)* $(\forall x, y \in \mathcal{X})(x \circ y, y \in S \Rightarrow x \in S)$, *and (I2)* $(\forall x \in S, y \in \mathcal{X})(x \circ y \in S)$.

(3) *normal of* $\mathcal{X}$ *if* $(\forall x, y, a, b \in \mathcal{X})(x \circ y, a \circ b \in S \Rightarrow (x \circ a) \circ (y \circ b) \in S)$.

We know that $\{0\}$ is a Bd-subalgebra but it is not a Bd-ideal of $\mathcal{X}$. For example, in Example 2.4, $0 \circ b = c \notin \{0\}$. The set $\{0\}$ does not satisfy (I2).

Example 2.11. *Let* $\mathcal{X} = \{0, a, b, c\}$ *be a set with a binary operation* $\circ$ *defined by the following Cayley table:*

$\circ$	0	a	b	c
0	0	a	a	a
a	a	a	a	a
b	b	b	b	b
c	c	a	a	a

Then $\mathcal{X} = (\mathcal{X}, \circ, 0)$ *is a Bd-algebra and* $\{0, a, c\}$ *is a Bd-ideal of* $\mathcal{X}$.

Proposition 2.12. *If* $\mathcal{X}$ *is edge, then* $\{0\}$ *is a Bd-ideal of* $\mathcal{X}$.

Proof. As (Bd1), we have $\{0\}$ satisfies (I1). Since $\mathcal{X}$ is edge, we have $0 \circ x = 0$ for all $x \in \mathcal{X}$, implying that $\{0\}$ satisfies (I2). Hence, $\{0\}$ is a Bd-ideal of $\mathcal{X}$. ☐

Example 2.13. *In Example 2.4,* $S = \{0, a\}$ *is a Bd-subalgebra of* $\mathcal{X}$. *In addition, we have* S *is normal. However, it is not a Bd-ideal.*

The following proposition follows directly from (I2).

Proposition 2.14. *If* $\{0\}$ *is a Bd-ideal of* $\mathcal{X}$, *then* $0 \circ x = 0$, *for all* $x \in \mathcal{X}$.

The following proposition follows directly from (Bd1).

Proposition 2.15. *Every normal subset of* $\mathcal{X}$ *is a Bd-subalgebra.*

Lemma 2.16. *If* $\mathcal{X}$ *satisfies (Bd8)* $(\forall x, y, a, b \in \mathcal{X})((x \circ y) \circ (a \circ b) = (y \circ a) \circ (x \circ b))$, *then it is commutative.*

Proof. Let $x, y \in \mathcal{X}$. Then $y \circ x = (y \circ x) \circ 0 = (y \circ x) \circ (0 \circ 0) = (x \circ 0) \circ (y \circ 0) = x \circ y$. Hence, $\mathcal{X}$ is commutative. ☐

Let $\mathcal{X}$ satisfy (B1) and (Bd8) and let S be a Bd-subalgebra of $\mathcal{X}$. We define a relation $\bowtie_S$ on $\mathcal{X}$ by

$$(\forall x, y \in \mathcal{X})(x \bowtie_S y \Leftrightarrow x \circ y \in S).$$

Reflexivity: Let $x \in \mathcal{X}$. By (B1), we have $x \circ x = 0 \in S$. Thus $x \bowtie_S x$.

Symmetry: Let $x, y \in \mathcal{X}$ be such that $x \bowtie_S y$. Then $x \circ y \in S$. By Lemma 2.16, we have $y \circ x \in S$. Thus $y \bowtie_S x$.

Transitivity: Let $x, y, z \in \mathcal{X}$ be such that $x \bowtie_S y$ and $y \bowtie_S z$. Then $x \circ y \in S$ and $y \circ z \in S$. By Lemma 2.16, we have $z \circ y \in S$. By (B1) and (Bd8), we have $x \circ z = (x \circ z) \circ 0 = (x \circ z) \circ (y \circ y) = (z \circ y) \circ (x \circ y) \in S$. Thus $x \bowtie_S z$.

Compatible: Let $x, y, z \in \mathcal{X}$ be such that $x \bowtie_S y$. Then $x \circ y \in S$. By Lemma 2.16, we have $y \circ x \in S$. By (B1) and (Bd8), we have $(x \circ z) \circ (y \circ z) = (y \circ x) \circ (z \circ z) = (y \circ x) \circ 0 = y \circ x \in S$. Thus $x \circ z \bowtie_S y \circ z$. By Lemma 2.16, we have $z \circ x \bowtie_S z \circ y$.

Therefore, $\bowtie_S$ is a congruence relation on $\mathcal{X}$. We denote the equivalence class containing x by $\ulcorner x \lrcorner_S$ and $\mathcal{X}/S := \{\ulcorner x \lrcorner_S : x \in \mathcal{X}\}$.

We can summarize the result in the following theorem.

Theorem 2.17. *Let $\mathcal{X}$ satisfy (B1) and (Bd8) and let S be a Bd-subalgebra of $\mathcal{X}$. Define a binary operation $\star$ on $\mathcal{X}/S$ by*

$$(\forall x, y \in \mathcal{X})(\ulcorner x \lrcorner_S \star \ulcorner y \lrcorner_S = \ulcorner x \circ y \lrcorner_S).$$

Then $(X/S, \star, \ulcorner 0 \lrcorner_S)$ is a Bd-algebra satisfying (B1) and (Bd8).

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