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Approximating Common Fixed Points of α-Nonexpansive Mappings in CAT(0) Spaces

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Abstract In this paper, we prove and approximate common fixed points of two α-nonexpansive mappings through strong and Δ-convergence of an iterative sequence in a CAT(0) space. Moreover, we expand and improve the result of Muangchoo-in et al. [1].

MSC: 47H09; 47H10
Keywords: fixed point set; square α-nonexpansive mapping; iteration; CAT(0) spaces

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1. Introduction

Let (X,d) be a metric space and $x,y\in X$ with $l=d(x,y)$. A geodesic path from x to y is an isometry $\gamma:[0,d(x,y)]\to X$ such that $\gamma(0)=x$, $\gamma(d(x,y))=y$, and $d(\gamma(t_1),\gamma(t_2))=|t_1-t_2|$ for any $t_1,t_2\in[0,d(x,y)]$. We will say that (X,d) is a (uniquely) geodesic metric space if any two points are connected by a (unique) geodesic. In this case, we denote such geodesic by $[x,y]$. Note that in general such geodesic is not uniquely determined by its endpoints. For a point $z\in[x,y]$, we will use the notation $z=(1-t)x\oplus ty$, where $t=\frac{d(x,z)}{d(x,y)}$, $1-t=\frac{d(y,z)}{d(x,y)}$ assuming $x\neq y$. Let (X,d) be a geodesic metric space. A geodesic triangle consists of three point $p,q,r\in X$ and three geodesics $[p,q],[q,r],[r,p]$. Denote $\Delta([p,q],[q,r],[r,p])$. For such a triangle, there is a comparison triangle $\overline{\Delta}(\overline{p},\overline{q},\overline{r})\to\mathbb{E}^2$: $d(p,q)=d(\overline{p},\overline{q})$, $d(q,r)=d(\overline{q},\overline{r})$, $d(r,p)=d(\overline{r},\overline{p})$.

Definition 1.1. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
Cat(0): Let $\Delta=(x_1,x_2,x_3)$ be a geodesic triangle in b-metric space X and let $\bar{\Delta}\in\mathbb{E}^2$

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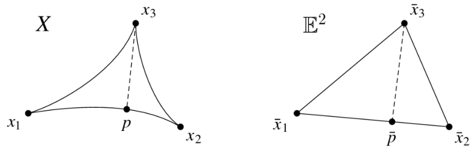
be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta} := (\overline{x_1}, \overline{x_2}, \overline{x_3})$ such that

$$d(x, y) \leq d^2_{\mathbb{E}}(\overline{x}, \overline{y}).$$



It is easy to see that a CAT(0) space is uniquely geodesic.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include inner product spaces, R-trees (see, for example, [2]), Euclidean building (see, for example, [3]), and the complex Hilbert ball with a hyperbolic metric (see, for example, [4]). For a thorough discussion on other spaces and on the fundamental role they play in geometry, see, for example, [2]-[12].

We collect some properties of CAT(0) spaces. For more details, we refer the readers to [13]-[15].

Lemma 1.2 ([13]). *Let (X, d) be a CAT(0) space. Then the following assertions hold.*
(i) For x, y in X and t in [0, 1], there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \tag{1.1}$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (1.1)
(ii) For x, y in X and t in $[0, 1]$, we have

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z). \tag{1.2}$$

Example 1.3. *(I).* Let $X := l_p(\mathbb{R})$ where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{i=1}^{\infty} |x_i| < \infty\}$. Define $d : X \times X \to [0, \infty)$ as:

$$d(x, y) = (\sum_{i=1}^{\infty} |x_i - y_i|)$$

where $x = \{x_n\}, y = \{y_n\}$. Then d is a metric space, see([16] -[18]). And, defined a continuous mapping $\gamma : [0, d(x, y)] \to X$ by $\gamma(z) = (1 - t)x + ty$ for all $t \in [0, d(x, y)]$. and all $z \in X$. Then (X, d) is a CAT(0) space.

(II). Let $X := L_p[0, 1]$ be the space of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|dt < \infty$. Define $d : X \times X \to [0, \infty)$ as:

$$||x|| = (\int_0^1 |x(t)|dt)$$

where $x = x(t)$. Then d is a metric space, see([16] -[18]). And, defined a continuous mapping $\gamma : [0, d(x, y)] \to X$ by $\gamma(z) = (1 - t)x + ty$ for all $t \in [0, d(x, y)]$. and all $z \in X$. Then (X, d) is a CAT(0) space.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center A $(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$$

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$, see [19].

Lemma 1.4 ([20]). *Every bounded sequence in a complete CAT(0) space X has a Δ-convergent subsequence.*

Lemma 1.5 ([21]). *Let C be a closed and convex subset of a complete CAT(0) space X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 1.6 ([22]). *Let X be a complete CAT(0) space and let $x \in X$. Suppose that $0 < b \leq t_n \leq c < 0$ and $x_n, y_n \in X$ for $n = 1, 2, \ldots$. If for some $r \geq 0$ we have*

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \quad \limsup_{n\to\infty} d(y_n, x) \leq r,$$

and $\lim_{n\to\infty} d(t_n x_n \oplus (1 - t_n)y_n, x) = r$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.7 ([23]). *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. If $\{x_n\}$ is a sequence in C such that $d(Tx_n, x_n) \to 0$ and $\Delta - \lim_{n\to\infty} x_n = z$ for some $z \in X$, then $z \in C$ and $Tz = z$.*

Now, we recall definitions of α-nonexpansive mappings in CAT(0).

Definition 1.8. (Aoyama and Kohsaka [24]) Let (X, d) be a metric space and C be nonempty subset. Then $T : C \to C$ said to be a square α-nonexpansive mapping (or α-noexpansive mapping), if $\alpha < 1$ such that

$$d^2(Tx, Ty) \leq \alpha d^2(Tx, y) + \alpha d^2(x, Ty) + (1 - 2\alpha)d^2(x, y),$$

for all $x, y \in C$.

Definition 1.9 ([1]). Let (X, d) be a metric space and C be nonempty subset. Then $T : C \to C$ said to be a quasi-nonexpansive if $F(T) \neq \emptyset$; and $d(Tx, p) \leq d(x, p)$ for all $p \in F(T); = \{x \in X | x = Tx\}$, and $x \in C$.

Lemma 1.10 ([1]). *Let C be a nonempty subset of a hyperbolic space X. Let $T : C \to C$ be a square α-nonexpansive mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

On the other hand, we recall that iterations in CAT(0) spaces. We begin the Ishikawa iteration in CAT(0) spaces is described as follows: For any initial point $x \in C$, we define the iterates $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \gamma_n y_n \oplus (1 - \gamma_n)x_n \\ y_n = \beta_n T x_n \oplus (1 - \beta_n)x_n \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$, see [25].

In 2018, Muangchoo-in, Kumam and Je Cho [26] introduced and approximated common fixed points of two alpha-nonexpansive mappings through weak and strong convergence of an iterative sequence in a uniformly convex Babach space.

$$\begin{cases} x_{n+1} = \gamma_n S y_n \oplus (1-\gamma_n) x_n \\ y_n = \beta_n T x_n \oplus (1-\beta_n) x_n \ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$.

In this paper, we prove and approximate common fixed points of two α-nonexpansive mappings through strong and Δ-convergence of an iterative sequence in a CAT(0) space. Moreover, we expand and improve the result of Muangchoo-in et al. [1].

2. Main Results

In this section, we state some useful lemmas as follows.

Lemma 2.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let $S, T : C \to C$ be square α-nonexpansive mappings and $F(S) \cap F(T)$ be a the set of all common fixed points of two nonexpansive mappings T and S of C. Assume there exists $p \in F(S) \cap F(T)$. Suppose that $\{x_n\}$ is defined by Ishikawa's iteration (1.3). Then*

$$\lim_{n\to\infty} d(Sx_n, x_n) = 0 = \lim_{n\to\infty} d(Tx_n, x_n).$$

Proof. Let $p \in F(S) \cap F(T)$. By Lemma 1.10 we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\gamma_n)x_n \oplus \gamma_n S y_n, p) \\ &\le (1-\gamma_n)d(x_n, p) + \gamma_n d(Sy_n, p) \\ &\le (1-\gamma_n)d(x_n, p) + \gamma_n d(y_n, p) \\ &= (1-\gamma_n)d(x_n, p) + \gamma_n d((1-\beta_n)x_n \oplus \beta_n T x_n, p) \\ &\le (1-\gamma_n)d(x_n, p) + \gamma_n(1-\beta_n)d(x_n, p) + \gamma_n\beta_n d(Tx_n, p) \\ &\le (1-\gamma_n)d(x_n, p) + \gamma_n(1-\beta_n)d(x_n, p) + \gamma_n\beta_n d(x_n, p) \\ &\le (1-\gamma_n)d(x_n, p) + \gamma_n(1-\beta_n)d(x_n, p) + \gamma_n\beta_n d(Tx_n, p) \\ &= d(Tx_n, p) \end{aligned}$$

Hence $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p) = r$ where $r = 0$ is a real number. By T is quasi-nonexpansive mapping then we have $d(Tx_n, p) \le d(x_n, p)$ for all $n = 1, 2, 3, \ldots$. So $\limsup_{n\to\infty} d(Tx_n, p) = \limsup_{n\to\infty} d(x_n, p) = r$. Also,

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p), \end{aligned} \tag{2.1}$$

and by S is quasi-nonexpansive mapping then we obtain that

$$\limsup_{n\to\infty} d(Sy_n, p) \le \limsup_{n\to\infty} d(y_n, p) \le r. \tag{2.2}$$

Moreover, $\lim_{n\to\infty} d(x_{n+1}, p) = r$ means that

$$\lim_{n\to\infty} d(\gamma_n Sy_n \oplus (1-\gamma_n)x_n, p) = r. \tag{2.3}$$

By Lemma 1.6 , we get that

$$\lim_{n\to\infty} d(Sy_n, x_n) = 0. \tag{2.4}$$

Since $d(x_n, p) \le d(x_n, Sy_n) + d(Sy_n, p) \le d(x_n, Syn) + d(y_n, p)$, then we obtain that

$$r \le \liminf_{n\to\infty} d(y_n, p) \tag{2.5}$$

By (2.2) and (2.5), we obtain that

$$\lim_{n\to\infty} d(\beta_n Tx_n \oplus (1-\beta_n)x_n, p) = \lim_{n\to\infty} d(y_n, p) = 0 \tag{2.6}$$

By Lemma 1.6 , we get that

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{2.7}$$

Noe, we consider

$$\begin{aligned} d(Tx_n, y_n) &= d(Tx_n, \beta_n Tx_n \oplus (1-\beta_n)x_n) \\ &\le (1-\beta_n)d(Tx_n, Tx_n) + \beta_n d(Tx_n, x_n) \\ &= \beta_n d(Tx_n, x_n), \end{aligned} \tag{2.8}$$

then by (2.7), we have

$$\lim_{n\to\infty} d(Tx_n, y_n) = 0. \tag{2.9}$$

By Definition 1.8, we consider

$$\begin{aligned} d(Sx_n, x_n)^2 \le& (d(Sx_n, Sy_n) + d(Sy_n, x_n))^2 \\ =& d(Sx_n, Sy_n)^2 + 2d(Sx_n, Sy_n)d(Sy_n, x_n)) + d(Sy_n, x_n))^2 \\ \le& \alpha d(Sx_n, y_n)^2 + \alpha d(x_n, Sy_n)^2 + (1-2\alpha)d(x_n, y_n)^2 \\ &+ 2d(Sx_n, Sy_n)d(Sy_n, x_n)) + d(Sy_n, x_n))^2 \\ \le& \alpha(d(Sx_n, x_n) + d(x_n, y_n))^2 + (1-2\alpha)d(x_n, y_n)^2 \\ &+ 2d(Sx_n, Sy_n)d(Sy_n, x_n)) + (1+\alpha)d(Sy_n, x_n))^2 \\ \le& \alpha d(Sx_n, x_n)^2 + \alpha 2d(Sx_n, x_n)d(x_n, y_n) + \alpha d(x_n, y_n)^2 \\ &+(1-2\alpha)d(x_n, y_n)^2 + 2d(Sx_n, Sy_n)d(Sy_n, x_n)) + (1+\alpha)d(Sy_n, x_n))^2, \end{aligned} \tag{2.10}$$

so

$$\begin{aligned} (1-\alpha)d(Sx_n, x_n)^2 \le& (1-\alpha)d(x_n, y_n)^2 + \alpha 2d(Sx_n, x_n)d(x_n, y_n) \\ &+ 2d(Sx_n, Sy_n)d(Sy_n, x_n)) + (1+\alpha)d(Sy_n, x_n))^2 \\ \le& (1-\alpha)(d(x_n, Tx_n) + d(Tx_n, y_n))^2 \\ &+ 2\alpha d(Sx_n, x_n)(d(x_n, Tx_n) + d(Tx_n, y_n)) \\ &+ 2d(Sx_n, Sy_n)d(Sy_n, x_n)) + (1+\alpha)d(Sy_n, x_n))^2 \end{aligned} \tag{2.11}$$

By (2.4), (2.7) and (2.9), we conclude that

$$\lim_{n\to\infty} d(Sx_n, x_n) = 0 = \lim_{n\to\infty} d(Tx_n, x_n).$$

■

Theorem 2.2. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let $S, T : C \to C$ be square α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be Ishikawa's iteration. If $F(S) \cap F(T) \neq \emptyset$ then then $\{x_n\}$ Δ-converges to a unique common fixed point of S and T.*

Proof. Let p be a common fixed point of S and T and $\lim_{n\to\infty} d(x_n, p)$ exists. Thus $\{x_n\}$ is bounded. Therefore $\{x_n\}$ has a Δ-convergent subsequence and the asymptotic center of $\{x_n\}$ is in C by Lemma 1.4, 1.5. We now prove that every Δ-convergent subsequence of $\{x_n\}$ has a unique Δ-limit in $F(S) \cap F(T)$. For, let u and v be two Δ-limits of the subsequences $\{u_n\}$ and $\{v_n\}$ of $\{x_n\}$, respectively. By definition $A(\{u_n\}) = \{u\}$ and $A(\{v_n\}) = \{v\}$. By Lemma 2.1, $\lim_{n\to\infty} d(Su_n, u_n) = 0 = \lim_{n\to\infty} d(Tu_n, u_n)$. Now using the Δ-convergence of $\{u_n\}$ to u and the α-nonexpansive mappings of T and S, we obtain $u \in F(S) \cap F(T)$ by a repeated application of Lemma 1.7 on T and S. Again in the same fashion, we can prove that $v \in F(S) \cap F(T)$. Next, we prove the uniqueness. To this end, if u and v are distinct then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, u) &= \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v) \\
&< \limsup_{n\to\infty} d(v_n, u) \\
&= \limsup_{n\to\infty} d(x_n, u) \\
&= \lim_{n\to\infty} d(x_n, u). \qquad (2.12)
\end{aligned}$$

This is a contradiction, then $u = v$. ■

Theorem 2.3. *Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let $S, T : C \to C$ be square α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be Ishikawa's iteration. If $F(S) \cap F(T) \neq \emptyset$ then $\{x_n\}$converges strongly to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F(S) \cap F(T)) = 0$, where $d(x, F(S) \cap F(T)) := \inf\{d(x, p) | p \in F(S) \cap F(T)\}$.*

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(S) \cap F(T)) = 0$. As proved in Lemma 2.1, we have

$$d(x_{n+1}, p) \leq d(x_n, p), \text{ for all } p \in F(S) \cap F(T).$$

This implies that $d(x_{n+1}, F(S) \cap F(T)) \leq d(x_n, F(S) \cap F(T))$, so that $d(x_n, F(S) \cap F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(S) \cap F(T)) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(S) \cap F(T)) = 0$, there exists a positive integer n_0 such that $d(x_n, F(S) \cap F(T)) < \frac{\epsilon}{4}, \forall n \geq n_0$. In particular, $\inf\{d(x_{n_0}, p) | p \in F(S) \cap F(T)\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F(S) \cap F(T)$ such that $d(x_{n_0}, p^*) < \frac{\epsilon}{2}$. Now, for all $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(p^*, x_n) \leq 2d(x_{n_0}, p^*) < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete CAT (0) space, and so it must converge to a point p in C. Now, $\lim_{n\to\infty} d(x_n, F(S) \cap F(T)) = 0$ gives that $d(p, F(S) \cap F(T)) = 0$. Since F is closed, so we have $p \in F(S) \cap F(T)$. ∎

3. Numerical Example

In this section, we provide some numerical examples and illustrate its performance by using algorithm 1.4.

Example 3.1. Let $X := \mathbb{R}$ with metric $d(x, y) = |x - y|$ and $C = [0, 1]$. Define $\gamma : [0, d(x, y)] \to X$ by $\alpha x \oplus (1-\alpha)y := \alpha x + (1-\alpha)y$ for each $x, y \in X$ and $\alpha \in [0, 1]$. We can proof that (X, d) is a complete CAT(0) space and C is a nonempty closed convex subset of X. For a given $m \in (0, 1)$, let $T : C \to C$ defined by

$$Tx := \begin{cases} \frac{x}{5} & \text{if } x \neq 1, \\ \frac{2+m}{5+m} & \text{if otherwise,} \end{cases}$$

and

$$Sx = \tfrac{x}{10} \text{ for all } x \in C.$$

We can easily prove that T and S are a square α-nonexpansive mapping. We set $m = 2$, $\beta_n = \frac{1}{12n+1}$ and $\gamma_n = \frac{1}{\sqrt{2n+1}}$, for all $n \geq 0$, we have

$$\begin{cases} x_{n+1} = \gamma_n S y_n + (1-\gamma_n) x_n \\ y_n = \beta_n T x_n + (1-\beta_n) x_n \ n \in \mathbb{N}, \end{cases} \tag{3.1}$$

The stopping criterion is defined by $E_n = |x_n - 0| \leq 10^{-6}$, where 0 is a fixed point of T. The numerical experiments, using our algorithm 1.4 for each choice x_0 are reported by using MATLAB in the following Table 1.

Table 1. Algorithm 1.4 with different choices of x_0

x_0	No. of Iter.	x_n
$x_0 = 0.01$	55	9.4649e-07
$x_0 = 0.2$	94	9.4821e-07
$x_0 = 0.4$	104	9.8232e-07
$x_0 = 0.6$	111	9.4766e-07
$x_0 = 0.8$	115	9.8815e-07
$x_0 = 0.99$	119	9.6047e-07

We concluded from Table 1 and Figure 1 that Ishikawa's iteration process is stable with respect to the choice of small value in C and parameters of the Table 1 also we observation that average number of iterations of the Ishikawa's iteration process is below respect to others processes.

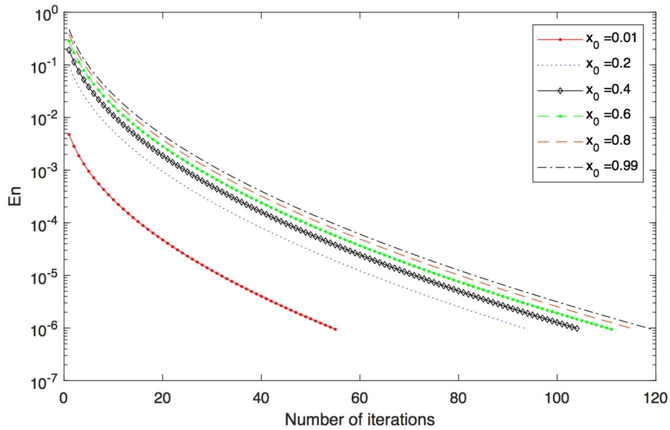


Figure 1. Error plotting E_n for Choice x_0 in Example 3.1.

4. Conclusion

In this paper, we establish results as follows:

(1) Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let $S, T : C \to C$ be square α-nonexpansive mappings and $F(S) \cap F(T)$ be a the set of all common fixed points of two nonexpansive mappings T and S of C. Assume there exists $p \in F(S) \cap F(T)$. Suppose that $\{x_n\}$ is defined by Ishikawa's iteration (1.3). Then

$$\lim_{n\to\infty} d(Sx_n, x_n) = 0 = \lim_{n\to\infty} d(Tx_n, x_n).$$

(2) Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let $S, T : C \to C$ be square α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be Ishikawa's iteration. If $F(S) \cap F(T) \neq \emptyset$ then then $\{x_n\}$ Δ-converges to a unique common fixed point of S and T.

(3) Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d). Let $S, T : C \to C$ be square α-nonexpansive mappings. Assume C satisfies Opial's condition and the sequence defined be Ishikawa's iteration. If $F(S) \cap F(T) \neq \emptyset$ then $\{x_n\}$converges strongly to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F(S) \cap F(T)) = 0$, where $d(x, F(S) \cap F(T)) := \inf\{d(x, p)|p \in F(S) \cap F(T)\}$.

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