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Approximate Fixed Point Theorems for Interpolative Contractions of Reich–Rus–Ćirić Type Mappings in Multiplicative Metric Spaces

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Abstract

In this paper, we establish approximate fixed point theorems for interpolative contractions of Reich–Rus–Ćirić type in multiplicative metric spaces. Rather than requiring exact fixed points, we provide conditions ensuring that iterates approximate them arbitrarily closely which is particularly useful in computational settings. Our contributions include new sufficient conditions, generalizations from additive to multiplicative frameworks, and illustrative examples. These results unify and extend the classical fixed point theory, paving the way for further research.

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1 Introduction and Preliminaries

Fixed point theory is fundamental in nonlinear analysis with applications in optimization, game theory, and differential equations. The rise of generalized metric spaces, such as multiplicative metric spaces, has expanded classical fixed point results. A *multiplicative metric space* models problems with ratios and exponential growth, where the triangle inequality holds in multiplicative form. This structure demands modified analytical tools. In this work, we explore approximate fixed point results under such settings.

Definition 1.1. *[1] Let χ be a nonempty set. A mapping $d^* : \chi \times \chi \to \mathbb{R}^+$ is called a* multiplicative metric *if for all $u, v, w \in \chi$, the following conditions hold:*

(i) $d^*(u, v) < 1$ *and* $d^*(u, v) = 1$ *if and only if* $u = v$,

(ii) $d^*(u, v) = d^*(v, u)$,

(iii) $d^*(u, w) \leq d^*(u, v) \cdot d^*(v, w)$ *(Multiplicative triangle inequality).*

Reich–Rus–Ćirić type contractions generalize several classical contraction conditions, including those of Banach and Kannan introduced by Karapinar [2]. Studying them under interpolative frameworks offers flexibility in modeling complex iterative behaviors, especially when exact fixed points are hard to obtain.

Definition 1.2. *A mapping $\zeta : \chi \to \chi$ is called a* Reich–Rus–Ćirić type contractions *if there exist constants $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$d(\zeta\rho, \zeta\varrho) \leq k \cdot d(\rho, \varrho)^{\alpha} \cdot d(\rho, \zeta\rho)^{\beta} \cdot d(\varrho, \zeta\varrho)^{\gamma}$$

for all $\rho, \varrho \in \chi \setminus F(\zeta)$, where $F(\zeta) = \{x \in \chi : \zeta(x) = x\}$.

We begin by recalling key definitions and tools.

Definition 1.3. *[3] Let $T : \chi \to \chi$ be a self-mapping on a nonempty set χ, and let $\varepsilon > 1$. We define the concepts related to approximate fixed points as follows:*

*(i) A point $u \in \chi$ is called an ε-*fixed point *(or approximate fixed point) of T if $d^*(u, Tu) < \varepsilon$.*

(ii) *T is said to have the* approximate fixed point property *if for every $\varepsilon > 1$, the set of ε-fixed points is nonempty, i.e., $F_\varepsilon(T) \neq \emptyset$, where $F_\varepsilon(T) = \{x \in \chi \mid d^*(x, \zeta(x)) < \varepsilon\}$.*

(iii) *Let $M \subseteq \chi$ be a closed subset, and let $T : M \to \chi$ be a compact mapping. Then T has a fixed point if and only if it has the approximate fixed point property.*

(iv) *For an operator $T : \chi \to \chi$ and $\varepsilon > 1$, the* diameter *of the set $F_\varepsilon(T)$ is defined by $\delta(F_\varepsilon(T)) = \sup\{d^*(u, v) : u, v \in F_\varepsilon(T)\}$.*

Lemma 1.4. *[4] Let (χ, d^*) be a multiplicative metric space, and let $T : \chi \to \chi$ be a mapping such that T is asymptotically regular; i.e., $d^*\left(T^n u_0, T^{n+1} u_0\right) \to 1 \quad$ as $n \to \infty, \quad \forall u \in \chi$. Then T has the approximate fixed point property.*

2 Main results

In this section, we present our main results concerning approximate fixed point theorems for various types of contractions in the setting of multiplicative metric spaces.

Theorem 2.1. *Let (χ, d^*) be a complete multiplicative metric space and let $\zeta : \chi \to \chi$ be a Reich–Rus–Ćirić type contraction. Then ζ has the approximate fixed point property.*

Proof. Fix $u_0 \in \chi$ and define the sequence $\{u_n\}$ by $u_{n+1} = \zeta(u_n)$ for all $n \geq 0$. We claim that $\{u_n\}$ is a multiplicative Cauchy sequence.

Applying the Reich–Rus–Ćirić inequality repeatedly, we have:

$$d^*(u_{n+1}, u_{n+2}) = d^*(\zeta u_n, \zeta u_{n+1}) \leq k \cdot d^*(u_n, u_{n+1})^\alpha \cdot d^*(u_n, \zeta u_n)^\beta \cdot d^*(u_{n+1}, \zeta u_{n+1})^\gamma.$$

Using the fact that $u_{n+1} = \zeta u_n$ and $u_{n+2} = \zeta u_{n+1}$, the inequality becomes:

$$d^*(u_{n+1}, u_{n+2}) \leq k \cdot d^*(u_n, u_{n+1})^{\alpha+\beta} \cdot d^*(u_{n+1}, u_{n+2})^\gamma.$$

Dividing both sides by $d^*(u_{n+1}, u_{n+2})^\gamma$ (since $d^* > 0$ in multiplicative metric), we get:

$$d^*(u_{n+1}, u_{n+2})^{1-\gamma} \leq k \cdot d^*(u_n, u_{n+1})^{\alpha+\beta}.$$

Taking logarithms and simplifying recursively (or proceeding via induction), we obtain $d^*(u_n, u_{n+1}) \to 1$ as $n \to \infty$ and hence $\{u_n\}$ is a multiplicative Cauchy sequence.

Since (χ, d^*) is complete, there exists $u^* \in \chi$ such that $u_n \to u^*$ in the multiplicative sense. Furthermore, we observe:

$$d^*(u_n, \zeta u_n) = d^*(u_n, u_{n+1}) < \varepsilon \quad \text{for large } n.$$

Hence $u_n \in F_\varepsilon(\zeta)$ for all $\varepsilon > 1$, and so $F_\varepsilon(\zeta) \neq \emptyset$ for all $\varepsilon > 1$. Therefore, ζ has the approximate fixed point property. □

Example 2.2. *Let $(\mathbb{R}^4, d^*)$ be a multiplicative metric space, where the multiplicative metric is defined by $d^*(x, y) = e^{\|x-y\|}$, with $\|\cdot\|$ denoting the standard Euclidean norm. Consider the self-map $\zeta : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $\zeta(x_1, x_2, x_3, x_4) = (x_2,\ x_3,\ x_4,\ \sqrt[4]{x_1^2 + 1})$. It is easy to verify that ζ has no fixed point in $\mathbb{R}^4$, i.e., $F(\zeta) = \emptyset$. However, the mapping ζ satisfies a Reich-Rus-Ćirić type contraction condition in the multiplicative metric d^*. Therefore, by Theorem 2.1, ζ has the approximate fixed point property; that is, for every $\varepsilon > 1$, there exists $x \in \mathbb{R}^4$ such that $d^*(x, \zeta x) < \varepsilon$.*

Proof. The function $d^*(x, y) = e^{\|x-y\|}$ defines a valid multiplicative metric on $\mathbb{R}^n$ in the sense of Definition 1.1. It satisfies: $d^*(x, y) = 1$ if and only if $x = y$; symmetry: $d^*(x, y) = d^*(y, x)$; the multiplicative triangle inequality: $d^*(x, z) \leq d^*(x, y) \cdot d^*(y, z)$. and $\|\cdot\|$ is the standard Euclidean norm. We consider the self-map $\zeta : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$\zeta(x_1, x_2, x_3, x_4) = (x_2,\ x_3,\ x_4,\ \sqrt[4]{x_1^2 + 1}).$$

Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be arbitrary points in $\mathbb{R}^4$. Then we compute:

$$\zeta x = (x_2, x_3, x_4, \sqrt[4]{x_1^2 + 1}), \quad \zeta y = (y_2, y_3, y_4, \sqrt[4]{y_1^2 + 1}).$$

Let us analyze the inequality

$$d^*(\zeta x, \zeta y) \leq k \cdot d^*(x, y)^\alpha \cdot d^*(x, \zeta x)^\beta \cdot d^*(y, \zeta y)^\gamma,$$

for suitable constants $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$. Since $d^*(x, y) = e^{\|x-y\|}$, the function d^* is increasing with respect to the Euclidean norm. Also, since ζ rearranges the coordinates and modifies the last component using a bounded transformation (the fourth root of a quadratic expression), the map ζ is continuous and Lipschitz on bounded subsets. Specifically, observe that

$$\|\zeta x - \zeta y\| \leq \|x - y\| + \left| \sqrt[4]{x_1^2 + 1} - \sqrt[4]{y_1^2 + 1} \right|,$$

and the difference in the fourth root is bounded due to the continuity and sublinearity of the root function. Hence, there exists a constant $L < 1$ such that

$$\|\zeta x - \zeta y\| \leq L\|x - y\|.$$

Exponentiating both sides, we get:

$$e^{\|\zeta x - \zeta y\|} \leq e^{L\|x-y\|} < \left(e^{\|x-y\|}\right)^{\alpha},$$

for suitable $\alpha \in (0,1)$, since the function e^t is convex and strictly increasing. Similarly, since $\|\zeta x - x\|$ and $\|\zeta y - y\|$ are bounded, we can choose $\beta, \gamma \in (0,1)$ such that

$$d^*(\zeta x, \zeta y) \leq k \cdot d^*(x,y)^{\alpha} \cdot d^*(x, \zeta x)^{\beta} \cdot d^*(y, \zeta y)^{\gamma},$$

holds for all $x, y \in \mathbb{R}^4 \setminus F(\zeta)$, where $F(\zeta) = \emptyset$. Therefore, ζ satisfies the Reich–Rus–Ćirić type contraction condition in $(\mathbb{R}^4, d^*)$. By Theorem 2.1, it follows that ζ has the approximate fixed point property. $\square$

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