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Finite-time stability criteria of linear system with non-differentiable time-varying delay via new integral inequality

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Highlights

- A new integral inequality for bounding an integral in Lyapunov is proposed.
- Improved criteria guaranteeing FTS of linear system with time-varying are proposed.
- FTS criteria based on different forms of LKF are compared.
- The proposed criteria are practicable for continuous and non-differentiable delays.
- Criteria with new integral inequality yields better results for FTS with constant delay.

Abstract

In this article, a new integral inequality based on a free-matrix for bounding the integral $\int_a^b \dot{x}^T(u)R\dot{x}(u)du$ has been proposed. The new inequality and appropriated Lyapunov–Krasovskii functional play key roles for deriving finite-time stability criteria of linear systems with constant and continuous non-differentiable time-varying delays. The new sufficient finite-time stability conditions have been proposed in the forms of inequalities and linear matrix inequalities. In addition, we apply the same procedure as done for deriving finite-time stable criteria but using Wirtinger-based inequality instead of our new inequality and compare these criteria with other works. At the end, two numerical examples are presented to show that the proposed criteria are practicable for linear systems with non-differentiable delay. Criteria using proposed integral inequality yield better results than the other works for linear system with constant delay. However, results using Wirtinger inequality are less conservative when time-varying delay is considered.

Keywords: Finite-time stability; Lyapunov–Krasovskii; Integral inequality; Wirtinger-based inequality

1. Introduction

Studies of real world problems in the fields of engineering or sciences related to atmospheric flow, chemical or population dynamics, such as forest insects or whales, are often described in the forms of a system of differential

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equations [6,12,14]. In many systems, considering long time behaviors of the state variables is not enough because values of state variables during the transient period may be too large or unrealistic before they reach their equilibrium. In chemical process, for example, temperature inside a container mixture must be maintained within a certain threshold during a fixed period of time in order that chemical substances can be in effect. This situation is commonly known as finite-time stability (FTS) which was introduced in 1961 by Dorato [9]. Therefore, many researchers have paid more attention to study the FTS of various systems. In the past few decades, researchers proposed criteria that guarantee FTS of various systems with finding the smallest upper bound of the norm square of state variables or finding maximum time that guarantees values of the state variables to be within the given bounds for a certain time. Some examples of FTS of linear system with constant delay are studied in [1–3,7,8,23,26]; FTS of linear system with time-varying delays [17,18,21,31]; and FTS on other systems (see [4,8–10,15,16,27]). To authors best knowledge, the past studies of FTS on linear system with time-varying delay are mostly limited to the delay in the form of differentiable functions. Thus, this article is devoted to the linear system with continuous time-varying delay that may be non-differentiable.

Studying the atmospheric dynamics, for example, researchers often used a model to see the development of such important behaviors. However, these behaviors have been affected by propagation delay in the atmosphere and may lead to mismodeling. Thus, delay is one of the main subjects that modelers must have been taken into account [19]. It is well-known from the past studies of stability of dynamical systems with delay that a small change in time-delay could imbalance the systems. Therefore, researchers have been trying to improve new sufficient criteria that guarantee systems to be stable with larger value of delays. There are several methods used to guarantee stability of the systems. One of the common methods is formulated via Lyapunov–Krasovskii functional (LKF) which is widely used to investigate the stability of the systems with delays or uncertainties.

In formulating a less conservative stability criterion, several complex LKFs in the form of multiple integral have been constructed. The derivative of LKFs is often found in the form of a single integral $\int_a^b x^T(s)Rx(s)ds$ or multiple integrals. Thus, bounding techniques are needed to bound these integrals. Several bounding techniques have been carried out in the past few decades. Examples of well-known inequalities are free-matrix-based [13,22], Moon's et al. inequality [20], Jensen's inequality [11] and Wirtinger-based integral inequality [24]. However, there is still some room for tightening the bounds of these integral inequalities. Some effort has been made recently to improve the bounds of these inequalities. Liu et al. [18] proposed a generalized version of Jensen's inequality over infinite domain; Zeng et al. [28] modified Wirtinger-based integral inequality and proposed a free-matrix-based inequality with 7 free matrices. Zhang et al. [29] extended Wirtinger's inequality and proposed an integral inequality that not only used information of state variables at both ends but also the information of single and double integrals of the state variables; i.e. $\int_a^b x^T(s)ds$ and $\int_a^b \int_a^s x^T(u)duds$. These modifications yield less conservative stability conditions compared to the aforementioned inequalities when the same LKF is chosen in some examples.

As mentioned above, FTS is one of the important topics that should have been further studied. Thus, in this article, we investigate the FTS of linear systems with constant and continuous time-varying delay functions. This article is organized as follow. In Section 2, we introduce the considered systems and review important definition and lemmas. Then, proof of the new integral inequality in the form of one free matrix is proposed. This inequality will be used for bounding the derivative of LKF which allows us to obtain delay dependent FTS criteria in Section 3. Two numerical examples are given in Section 4 to show the effectiveness of the proposed criteria. Conclusion is drawn in Section 5.

Notations: The following notations will be used throughout this article. $\mathbb{R}^n$ is the n-dimensional space with the scalar product x^Ty; $\mathbb{R}^{n\times m}$ denotes real value matrix with dimension $n \times m$; A^T denotes the transpose of the matrix A; $\lambda(A)$ are eigenvalues of A; $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) maximum (minimum) real part of $\lambda(A)$; $\|x\|$ represents Euclidean norm of vector x; $x_t(\theta) := x(t+\theta), \theta \in [-h_2, 0]$; $\|x_t(\theta)\|_s := \sup_{\theta\in[-h_2,0]}\{\|x(t+\theta)\|\}$; $A > 0$ (< 0) means A is positive (negative) definite; lower entries of symmetric matrix are represented by $*$.

2. Preliminaries

Consider linear system with interval time-varying delay

$$\dot{x}(t) = A_0x(t) + A_1x(t - h(t)), \tag{1}$$

where $t > 0, x(t) \in \mathbb{R}^n$ is the state vector of the system. $A_0, A_1 \in \mathbb{R}^{n\times n}$ are known constant matrices. The delay $h(t)$ is a continuous function satisfying

$$0 < h_1 \le h(t) \le h_2,\ \ h_1 \ne h_2. \tag{2}$$

We assume the initial condition as $x_0(\theta) = \phi(\theta)$, $\forall\theta \in [-h_2, 0]$ where the initial condition $\phi(\cdot)$ is a differentiable vector-valued function.

To formulate the FTS condition, we begin with introducing well-known definition and lemmas that will be used in the main theorems.

Definition 1 ([17])**.** Given a positive matrix U and three positive constants c_1, c_2, T with $c_1 < c_2$, the time-delay system described by Eq. (1) and delay condition as in Eq. (2) is said to be finite-time stable with respect to $(c_1, c_2, T, h_1, h_2, U)$, if the state variables satisfy the relationship:

$$\sup_{-h_2 \leq s \leq 0} \left\{ x^T(s)Ux(s), \dot{x}^T(s)U\dot{x}(s) \right\} \leq c_1 \Rightarrow x^T(t)Ux(t) < c_2, \quad \forall t \in [0, T].$$

Similarly, the linear system (1) with constant delay $h(t) = h$ is said to be finite-time stable with respect to (c_1, c_2, T, h, U), if the state variables satisfy

$$\sup_{-h \leq s \leq 0} \left\{ x^T(s)Ux(s), \dot{x}^T(s)U\dot{x}(s) \right\} \leq c_1 \Rightarrow x^T(t)Ux(t) < c_2, \quad \forall t \in [0, T].$$

Lemma 2 (*[5] Schur Complement Lemma*)**.** *Given constant matrices X, Y, Z with appropriate dimensions satisfying $Y = Y^T > 0$, $X = X^T$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -Y & Z \\ Z^T & X \end{bmatrix} < 0.$$

Lemma 3 (*[24] Wirtinger-based Integral Inequality*)**.** *For any symmetric constant matrix $R > 0$ and a differentiable signal $x : [a, b] \to \mathbb{R}^n$. The following inequality holds:*

$$\int_a^b \dot{x}^T(u)R\dot{x}(u)du \geq \frac{1}{b-a}\zeta^T \begin{bmatrix} 4R & 2R & -6R \\ * & 4R & -6R \\ * & * & 12R \end{bmatrix} \zeta,$$

where $\zeta = [x^T(b) \quad x^T(a) \quad \frac{1}{b-a}\int_a^b x^T(u)du]^T$.

Proposition 4. *For any positive definite matrix R, any differentiable function $x : [a, b] \to \mathbb{R}^n$. The following inequality holds:*

$$\int_a^b \dot{x}^T(u)R\dot{x}(u)du \geq \frac{1}{6(b-a)}\zeta^T \begin{bmatrix} 22R & 10R & -32R \\ * & 16R & -26R \\ * & * & 58R \end{bmatrix} \zeta,$$

where $\zeta = \begin{bmatrix} x^T(b) & x^T(a) & \frac{1}{b-a}\int_a^b x^T(u)du \end{bmatrix}^T$.

Proof. For any differentiable function $x(t) \in \mathbb{R}^n$, it is easy to see that

$$\int_a^b (a + b - 2u)\dot{x}(u)du = 2\int_a^b x(u)du - (b-a)\left[x(b) + x(a)\right], \tag{3}$$

$$\int_a^b (a + 2b - 3u)\dot{x}(u)du = 3\int_a^b x(u)du - (b-a)\left[x(b) + 2x(a)\right], \tag{4}$$

$$\int_a^b (a + b - 2u)du = 0, \tag{5}$$

$$\int_a^b (a+2b-3u)du = \frac{(b-a)^2}{2}, \tag{6}$$

$$\int_a^b (a+b-2u)^2 du = \frac{(b-a)^3}{3}, \tag{7}$$

$$\int_a^b (a+b-2u)(a+2b-3u)du = \frac{(b-a)^3}{2}, \tag{8}$$

$$\int_a^b (a+2b-3u)^2 du = (b-a)^3. \tag{9}$$

Define $\pi_1 = x(b) - x(a)$, $\pi_2 = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(u)du$, and $\pi_3 = x(b) + 2x(a) - \frac{3}{b-a}\int_a^b x(u)du$ and let

$$z(u) = \dot{x}(u) - \frac{1}{b-a}\pi_1 + \frac{(a+b-2u)}{(b-a)^2}\pi_2 + \frac{(a+2b-3u)}{(b-a)^2}\pi_3.$$

It is easy to see that $z^T(u)Rz(u) \geq 0$. Integrating this inequality from a to b, then applying Eqs. (3)–(9), we have

$$\begin{aligned}
0 &\leq \int_a^b z^T(u)Rz(u)du \\
&= \int_a^b \dot{x}^T(u)R\dot{x}(u)du - \frac{2}{b-a}\pi_1^T R\int_a^b \dot{x}(u)du + \frac{2}{(b-a)^2}\pi_2^T R\int_a^b (a+b-2u)\dot{x}(u)du \\
&\quad + \frac{2}{(b-a)^2}\pi_3^T R\int_a^b (a+2b-3u)\dot{x}(u)du + \frac{1}{(b-a)^2}\pi_1^T R\pi_1\int_a^b du \\
&\quad - \frac{2}{(b-a)^3}\pi_1^T R\pi_2\int_a^b (a+b-2u)du - \frac{2}{(b-a)^3}\pi_1^T R\pi_3\int_a^b (a+2b-3u)du \\
&\quad + \frac{1}{(b-a)^4}\pi_2^T R\pi_2\int_a^b (a+b-2u)^2du + \frac{1}{(b-a)^4}\pi_3^T R\pi_3\int_a^b (a+2b-3u)^2du \\
&\quad + \frac{2}{(b-a)^4}\pi_2^T R\pi_3\int_a^b (a+b-2u)(a+2b-3u)du \\
&= \int_a^b \dot{x}^T(u)R\dot{x}(u)du - \frac{6}{6(b-a)}\pi_1^T R\pi_1 - \frac{6}{6(b-a)}\pi_1^T R\pi_3 - \frac{10}{6(b-a)}\pi_2^T R\pi_2 \\
&\quad + \frac{6}{6(b-a)}\pi_2^T R\pi_3 - \frac{6}{6(b-a)}\pi_3^T R\pi_3.
\end{aligned} \tag{10}$$

Rewriting $[\pi_1^T \ \pi_2^T \ \pi_3^T]^T$ as

$$\begin{bmatrix}\pi_1\\ \pi_2\\ \pi_3\end{bmatrix} = \begin{bmatrix} I & -I & 0\\ I & I & -2I\\ I & 2I & -3I\end{bmatrix}\zeta.$$

Rearranging the inequality (10), thus, the proof is complete.

Remark 5. Idea of proof in Proposition 4 is based on the idea of Seuret & Gouaisbaut [24]. The new inequality is in the form of a free-matrix based similar to the one proposed in [11,28,30]. Notice that conditions in [28] and [30] require six and two free matrices, respectively; while the inequalities stated in Lemma 3 and Proposition 4 only require one free matrix that means less variable need to be determined.

The use of the inequality in Proposition 4 plays a key role in formulating less conservative FTS conditions in this paper. Now we are ready to formulate the main results as follows.

3. Main results

This section we divide into two parts. First, the FTS of linear system with time-varying is derived. Then the FTS of linear system with constant delay is followed. To formulate the main results, we first define common parameters occurring in the derivation of FTS conditions. Let h_1, h_2, h, α be constants and

$$\gamma_1 = \frac{e^{\alpha h_1} - 1}{\alpha}, \quad \gamma_2 = \frac{e^{\alpha h_2} - 1}{\alpha}, \quad \gamma_3 = \frac{e^{\alpha h_1} - \alpha h_1 - 1}{\alpha^2}, \quad \gamma_4 = \frac{e^{\alpha h_2} - e^{\alpha h_1} + \alpha(h_1 - h_2)}{\alpha^2},$$
$$\gamma_5 = \frac{e^{\alpha h_2} - \alpha h_2 - 1}{\alpha^2}, \quad \gamma_6 = \frac{e^{\alpha h} - 1}{\alpha}, \quad \gamma_7 = \frac{e^{\alpha h} - \alpha h - 1}{\alpha^2}.$$

3.1. Finite-time of linear system with time-varying delay

In this section, we formulate criteria to guarantee the FTS of the linear system with time-varying delay satisfying Eqs. (1)–(2).

Theorem 6. *Given a matrix $U > 0$. The linear system (1) with time-varying delay $h(t)$ satisfying Eq. (2) is finite-time stable with respect to $(c_1, c_2, T, h_1, h_2, U), 0 \le c_1 < c_2$, if there exist positive scalar α, symmetric positive-definite matrices $P, Q_1, Q_2, R_1, R_2, R_3$ and $R_4 \in \mathbb{R}^{n\times n}$ satisfying*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & 0 & 0 & 0 & \Psi_{19} \\ * & \Psi_{22} & \Psi_{23} & \Psi_{24} & \Psi_{25} & \Psi_{26} & \Psi_{27} & 0 & 0 \\ * & * & \Psi_{33} & \Psi_{34} & 0 & \Psi_{36} & 0 & \Psi_{38} & 0 \\ * & * & * & \Psi_{44} & 0 & 0 & \Psi_{47} & \Psi_{48} & \Psi_{49} \\ * & * & * & * & \Psi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Psi_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Psi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Psi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Psi_{99} \end{bmatrix} < 0, \tag{11}$$

and

$$\frac{\theta_2 c_1}{\theta_1} \le c_2 e^{-\alpha T}, \tag{12}$$

where $h_{21} = h_2 - h_1,\ \theta_1 = \lambda_{min}(\widetilde{P})$ *and*

$$\widetilde{P} = U^{-1/2} P U^{-1/2}, \quad \widetilde{Q_i} = U^{-1/2} Q_i U^{-1/2},\ i = 1, 2, \quad \widetilde{R_j} = U^{-1/2} R_j U^{-1/2},\ j = 1, 2, 3, 4,$$
$$\theta_2 = \lambda_{max}(\widetilde{P}) + \gamma_1\lambda_{max}(\widetilde{Q_1}) + \gamma_2\lambda_{max}(\widetilde{Q_2}) + \gamma_3\lambda_{max}(\widetilde{R_1}) + \gamma_4\lambda_{max}(\widetilde{R_2}) + \gamma_4\lambda_{max}(\widetilde{R_3}) + \gamma_5\lambda_{max}(\widetilde{R_4}),$$
$$\Psi_{11} = PA_0 + A_0^T P + Q_1 + Q_2 + A_0^T[h_1R_1 + h_{21}R_2 + h_{21}R_3 + h_2R_4]A_0 - \alpha P - \frac{11R_1}{3h_1} - \frac{11R_4}{3h_2},$$
$$\Psi_{12} = -\frac{5R_1}{3}, \quad \Psi_{13} = A_0^T[h_1R_1 + h_{21}R_2 + h_{21}R_3 + h_2R_4]A_1 + PA_1,$$
$$\Psi_{14} = -\frac{5R_4}{3h_2}, \quad \Psi_{15} = \frac{16R_1}{3h_1}, \quad \Psi_{19} = \frac{16R_4}{3h_2}, \quad \Psi_{22} = -e^{\alpha h_1}Q_1 - \frac{8R_1}{3h_1} - \frac{11R_2}{3h_{21}} - \frac{11R_3}{3h_{21}},$$
$$\Psi_{23} = -\frac{5R_3}{3h_{21}}, \quad \Psi_{24} = -\frac{5R_2}{3h_{21}}, \quad \Psi_{25} = \frac{13R_1}{3h_1}, \quad \Psi_{26} = \frac{16R_3}{3h_{21}}, \quad \Psi_{27} = \frac{16R_2}{3h_{21}},$$
$$\Psi_{33} = -\frac{19R_3}{3h_{21}} + A_1^T[h_1R_1 + h_{21}R_2 + h_{21}R_3 + h_2R_4]A_1, \quad \Psi_{34} = -\frac{5R_3}{3h_{21}}, \quad \Psi_{36} = \frac{13R_3}{3h_{21}}, \quad \Psi_{38} = \frac{16R_3}{3h_{21}},$$
$$\Psi_{44} = -e^{\alpha h_2}Q_2 - \frac{8R_2}{3h_{21}} - \frac{8R_3}{3h_{21}} - \frac{8R_4}{3h_2}, \quad \Psi_{47} = \frac{13R_2}{3h_{21}}, \quad \Psi_{48} = \frac{13R_3}{3h_{21}}, \quad \Psi_{49} = \frac{13R_4}{3h_2},$$
$$\Psi_{55} = -\frac{29R_1}{3h_1}, \quad \Psi_{66} = -\frac{29R_3}{3h_{21}}, \quad \Psi_{77} = -\frac{29R_2}{3h_{21}}, \quad \Psi_{88} = -\frac{29R_3}{3h_{21}}, \quad \Psi_{99} = -\frac{29R_4}{3h_2}.$$

Proof. Define

$$\zeta_1(t) = \big[x^T(t) \quad x^T(t-h_1) \quad \frac{1}{h_1}\int_{t-h_1}^{t} x^T(s)ds\big]^T,$$
$$\zeta_2(t) = \big[x^T(t-h_1) \quad x^T(t-h_2) \quad \frac{1}{h_{21}}\int_{t-h_2}^{t-h_1} x^T(s)ds\big]^T,$$
$$\zeta_3(t) = \big[x^T(t-h_1) \quad x^T(t-h(t)) \quad \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x^T(s)ds\big]^T,$$
$$\zeta_4(t) = \big[x^T(t-h(t)) \quad x^T(t-h_2) \quad \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x^T(s)ds\big]^T,$$
$$\zeta_5(t) = \big[x^T(t) \quad x^T(t-h_2) \quad \frac{1}{h_2}\int_{t-h_2}^{t} x^T(s)ds\big]^T,$$
$$\eta(t) = \big[x^T(t) \quad x^T(t-h_1) \quad x^T(t-h(t)) \quad x^T(t-h_2) \quad \frac{1}{h_1}\int_{t-h_1}^{t} x^T(s)ds \quad \frac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x^T(s)ds$$
$$\frac{1}{h_{21}}\int_{t-h_2}^{t-h_1} x^T(s)ds \quad \frac{1}{h_2-h(t)}\int_{t-h_2}^{t-h(t)} x^T(s)ds \quad \frac{1}{h_2}\int_{t-h_2}^{t} x^T(s)ds\big]^T.$$

Choose the LKF of the form $V(x(t)) = \sum_{i=1}^{7} V_i(x(t))$, where

$$V_1(x(t)) = x^T(t)Px(t), \quad V_2(x(t)) = \int_{t-h_1}^{t} e^{\alpha(t-s)}x^T(s)Q_1x(s)ds, \quad V_3(x(t)) = \int_{t-h_2}^{t} e^{\alpha(t-s)}x^T(s)Q_2x(s)ds,$$
$$V_4(x(t)) = \int_{-h_1}^{0}\int_{t+s}^{t} e^{\alpha(t-\theta)}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta ds, \qquad V_5(x(t)) = \int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{\alpha(t-\theta)}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta ds,$$
$$V_6(x(t)) = \int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{\alpha(t-\theta)}\dot{x}^T(\theta)R_3\dot{x}(\theta)d\theta ds, \qquad V_7(x(t)) = \int_{-h_2}^{0}\int_{t+s}^{t} e^{\alpha(t-\theta)}\dot{x}^T(\theta)R_4\dot{x}(\theta)d\theta ds.$$

Taking the derivatives of $V_i(x(t))$, $i = 1, 2, 3 \ldots, 7$, along the solution of Eq. (1), we obtain

$$\dot{V}_1(x(t)) = 2x^T(t)P\dot{x}(t) = x^T(t)\left[PA_0 + A_0^TP\right]x(t) + 2x^T(t)PA_1x(t-h(t)),$$
$$\dot{V}_2(x(t)) = x^T(t)Q_1x(t) - e^{\alpha h_1}x^T(t-h_1)Q_1x(t-h_1) + \alpha V_2(x(t)),$$
$$\dot{V}_3(x(t)) = x^T(t)Q_2x(t) - e^{\alpha h_2}x^T(t-h_2)Q_2x(t-h_2) + \alpha V_3(x(t)),$$
$$\dot{V}_4(x(t)) = h_1\dot{x}^T(t)R_1\dot{x}(t) - \int_{t-h_1}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds + \alpha V_4(x(t)),$$
$$\dot{V}_5(x(t)) = h_{21}\dot{x}^T(t)R_2\dot{x}(t) - \int_{t-h_2}^{t-h_1} e^{\alpha(t-s)}\dot{x}^T(s)R_2\dot{x}(s)ds + \alpha V_5(x(t)),$$
$$\dot{V}_6(x(t)) = h_{21}\dot{x}^T(t)R_3\dot{x}(t) - \int_{t-h_2}^{t-h_1} e^{\alpha(t-s)}\dot{x}^T(s)R_3\dot{x}(s)ds + \alpha V_6(x(t)),$$
$$\dot{V}_7(x(t)) = h_2\dot{x}^T(t)R_4\dot{x}(t) - \int_{t-h_2}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R_4\dot{x}(s)ds + \alpha V_7(x(t)).$$

From Eq. (1), we can rewrite $\dot{x}^T(t)R_i\dot{x}(t)$, $i = 1, 2, 3, 4$ as

$$\dot{x}^T(t)R_i\dot{x}(t) = x^T(t)A_0^TR_iA_0x(t) + 2x^T(t)A_0^TR_iA_1x(t-h(t)) + x^T(t-h(t))A_1^TR_iA_1x(t-h(t)).$$

Combining $\dot{V}_i(x(t)), i = 1, 2, \ldots, 7$, we obtain

$$\begin{aligned}\dot{V}(x(t)) - \alpha V(x(t)) \leq\, & x^T(t)\Big\{A_0^TP + PA_0 + Q_1 + Q_2 + A_0^T[h_1R_1 + h_{21}R_2 + h_{21}R_3 + h_2R_4]A_0 - \alpha P\Big\}x(t)\\ & + 2x^T(t)\Big\{A_0^T[h_1R_1 + h_{21}R_2 + h_{21}R_3 + h_2R_4]A_1 + PA_1\Big\}x(t-h(t))\\ & - x^T(t-h_1)e^{\alpha h_1}Q_1x(t-h_1) - x^T(t-h_2)e^{\alpha h_2}Q_2x(t-h_2)\\ & + x^T(t-h(t))A_1^T[h_1R_1 + h_{21}R_2 + h_{21}R_3 + h_2R_4]A_1x(t-h(t))\end{aligned}$$

$$
\begin{aligned}
&- \int_{t-h_1}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds - \int_{t-h_2}^{t-h_1} e^{\alpha(t-s)}\dot{x}^T(s)R_2\dot{x}(s)ds \\
&- \int_{t-h_2}^{t-h_1} e^{\alpha(t-s)}\dot{x}^T(s)R_3\dot{x}(s)ds - \int_{t-h_2}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R_4\dot{x}(s)ds. \qquad (13)
\end{aligned}
$$

Define

$$
\Xi_i = \begin{bmatrix} 22R_i & 10R_i & -32R_i \\ * & 16R_i & -26R_i \\ * & * & 58R_i \end{bmatrix}, \quad i = 1, 2, 3, 4.
$$

Consider $-\int_{t-h_1}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds$ in Eq. (13). Here, $s \in [t - h_1, t]$, thus, we have $1 \le e^{\alpha(t-s)}$. Applying Proposition 4 to this integral, This leads to

$$
- \int_{t-h_1}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R_1\dot{x}(s)ds \le - \int_{t-h_1}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds \le -\frac{1}{6h_1}\zeta_1^T(t)\Xi_1\zeta_1(t). \qquad (14)
$$

Similarly, we apply same arguments to the other 3 integrals in Eq. (13). This yields

$$
- \int_{t-h_2}^{t-h_1} e^{\alpha(t-s)}\dot{x}^T(s)R_2\dot{x}(s)ds \le - \int_{t-h_2}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds \le -\frac{1}{6h_{21}}\zeta_2^T(t)\Xi_2\zeta_2(t), \qquad (15)
$$

$$
\begin{aligned}
- \int_{t-h_2}^{t-h_1} e^{\alpha(t-s)}\dot{x}^T(s)R_3\dot{x}(s)ds &\le - \int_{t-h_2}^{t-h_1} \dot{x}^T(s)R_3\dot{x}(s)ds \\
&= - \int_{t-h(t)}^{t-h_1} \dot{x}^T(s)R_3\dot{x}(s)ds - \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)R_3\dot{x}(s)ds \\
&\le -\frac{1}{6h_{21}}\zeta_3^T(t)\Xi_3\zeta_3(t) - \frac{1}{6h_{21}}\zeta_4^T(t)\Xi_3\zeta_4(t), \qquad (16)
\end{aligned}
$$

$$
- \int_{t-h_2}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R_4\dot{x}(s)ds \le - \int_{t-h_2}^{t} \dot{x}^T(s)R_4\dot{x}(s)ds \le -\frac{1}{6h_2}\zeta_5^T(t)\Xi_4\zeta_5(t). \qquad (17)
$$

Substituting relations in Eqs. (14)–(17) into Eq. (13), this yields $\dot{V}(x(t)) - \alpha V(x(t)) \le \eta^T(t)\Psi\eta(t)$, where Ψ is defined in Eq. (11). Because $\Psi < 0$, this leads to $\dot{V}(x(t)) - \alpha V(x(t)) < 0$. Multiplying this inequality by $e^{-\alpha t}$ and integrating from 0 to t with $t \in [0, T]$, we obtain

$$
e^{-\alpha t}V(x(t)) < V(x(0)),
$$

with

$$
\begin{aligned}
V(x(0)) = x^T(0)Px(0) &+ \int_{-h_1}^{0} e^{-\alpha s}x^T(s)Q_1x(s)ds + \int_{-h_2}^{0} e^{-\alpha s}x^T(s)Q_2x(s)ds \\
&+ \int_{-h_1}^{0}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta ds + \int_{-h_2}^{-h_1}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta ds \\
&+ \int_{-h_2}^{-h_1}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)R_3\dot{x}(\theta)d\theta ds + \int_{-h_2}^{0}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)R_4\dot{x}(\theta)d\theta ds.
\end{aligned}
$$

Since $I = U^{1/2}U^{-1/2} = U^{-1/2}U^{1/2}$, $\widetilde{P} = U^{-1/2}PU^{-1/2}$, $\widetilde{Q}_i = U^{-1/2}Q_iU^{-1/2}$, $i = 1, 2$, and $\widetilde{R}_j = U^{-1/2}R_jU^{-1/2}$, $j = 1, 2, 3, 4$, thus $V(x(0))$ can be written as

$$
\begin{aligned}
V(x(0)) = x^T(0)U^{1/2}\widetilde{P}U^{1/2}x(0) &+ \int_{-h_1}^{0} e^{-\alpha s}x^T(s)U^{1/2}\widetilde{Q}_1U^{1/2}x(s)ds \\
&+ \int_{-h_2}^{0} e^{-\alpha s}x^T(s)U^{1/2}\widetilde{Q}_2U^{1/2}x(s)ds + \int_{-h_1}^{0}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)U^{1/2}\widetilde{R}_1U^{1/2}\dot{x}(\theta)d\theta ds \\
&+ \int_{-h_2}^{-h_1}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)U^{1/2}\widetilde{R}_2U^{1/2}\dot{x}(\theta)d\theta ds + \int_{-h_2}^{-h_1}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)U^{1/2}\widetilde{R}_3U^{1/2}\dot{x}(\theta)d\theta ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-h_2}^{0}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)U^{1/2}\widetilde{R}_4U^{1/2}\dot{x}(\theta)d\theta ds\\
&\le c_1\Big[\lambda_{max}(\widetilde{P})+\lambda_{max}(\widetilde{Q}_1)\int_{-h_1}^{0}e^{-\alpha s}ds+\lambda_{max}(\widetilde{Q}_2)\int_{-h_2}^{0}e^{-\alpha s}ds+\lambda_{max}(\widetilde{R}_1)\int_{-h_1}^{0}\int_{s}^{0}e^{-\alpha\theta}d\theta ds\\
&\quad+\lambda_{max}(\widetilde{R}_2)\int_{-h_2}^{-h_1}\int_{s}^{0}e^{-\alpha\theta}d\theta ds+\lambda_{max}(\widetilde{R}_3)\int_{-h_2}^{-h_1}\int_{s}^{0}e^{-\alpha\theta}d\theta ds+\lambda_{max}(\widetilde{R}_4)\int_{-h_2}^{0}\int_{s}^{0}e^{-\alpha\theta}d\theta ds\Big]\\
&\le c_1\Big[\lambda_{max}(\widetilde{P})+\gamma_1\lambda_{max}(\widetilde{Q}_1)+\gamma_2\lambda_{max}(\widetilde{Q}_2)+\gamma_3\lambda_{max}(\widetilde{R}_1)+\gamma_4\lambda_{max}(\widetilde{R}_2)\\
&\quad+\gamma_4\lambda_{max}(\widetilde{R}_3)+\gamma_5\lambda_{max}(\widetilde{R}_4)\Big].
\end{aligned}
$$

Because $V(x(t)) \ge V_1(x(t)) = x^T(t)U^{1/2}\widetilde{P}U^{1/2}x(t) \ge \lambda_{min}(\widetilde{P})x^T(t)Ux(t)$. Thus, for any $t \in [0, T]$, we obtain

$$
\begin{aligned}
x^T(t)Ux(t) \le \frac{c_1e^{\alpha T}}{\lambda_{min}(\widetilde{P})}&\Big[\lambda_{max}(\widetilde{P})+\gamma_1\lambda_{max}(\widetilde{Q}_1)+\gamma_2\lambda_{max}(\widetilde{Q}_2)+\gamma_3\lambda_{max}(\widetilde{R}_1)\\
&+\gamma_4\lambda_{max}(\widetilde{R}_2)+\gamma_4\lambda_{max}(\widetilde{R}_3)+\gamma_5\lambda_{max}(\widetilde{R}_4)\Big] < c_2.
\end{aligned}
$$

Thus, the condition (12) holds and the proof is complete. □

Remark 7. The inequality (12) is not standard LMI. By defining

$$c_1\{\beta_2+\gamma_1\beta_3+\gamma_2\beta_4+\gamma_3\beta_5+\gamma_4\beta_6+\gamma_4\beta_7+\gamma_5\beta_8\} < c_2\beta_1e^{-\alpha T}.$$

Applying Schur's complement lemma as stated in Lemma 2 to this inequality, it is easy to verify that the feasibility of the inequality (12) is equivalent to the following relations and LMI:

$$\beta_1 I < \widetilde{P} < \beta_2 I,\ 0 < \widetilde{Q}_1 < \beta_3 I,\ 0 < \widetilde{Q}_2 < \beta_4 I, \tag{18}$$

$$0 < \widetilde{R}_1 < \beta_5 I,\ 0 < \widetilde{R}_2 < \beta_6 I,\ 0 < \widetilde{R}_3 < \beta_7 I,\ 0 < \widetilde{R}_4 < \beta_8 I, \tag{19}$$

$$\omega = \begin{bmatrix}
\omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} & \omega_{16} & \omega_{17} & \omega_{18}\\
* & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & -\beta_3 & 0 & 0 & 0 & 0 & 0\\
* & * & * & -\beta_4 & 0 & 0 & 0 & 0\\
* & * & * & * & -\beta_5 & 0 & 0 & 0\\
* & * & * & * & * & -\beta_6 & 0 & 0\\
* & * & * & * & * & * & -\beta_7 & 0\\
* & * & * & * & * & * & * & -\beta_8
\end{bmatrix} < 0, \tag{20}$$

where $I \in \mathbb{R}^{n\times n}$ is an identity matrix, $\omega_{11} = -\beta_1c_2e^{-\alpha T}$, $\omega_{12} = \beta_2\sqrt{c_1}$, $\omega_{13} = \beta_3\sqrt{c_1\gamma_1}$, $\omega_{14} = \beta_4\sqrt{c_1\gamma_2}$, $\omega_{15} = \beta_5\sqrt{c_1\gamma_3}$, $\omega_{16} = \beta_6\sqrt{c_1\gamma_4}$, $\omega_{17} = \beta_7\sqrt{c_1\gamma_4}$, $\omega_{18} = \beta_8\sqrt{c_1\gamma_5}$ for some positive scalars $\beta_i, i = 1, 2, \ldots, 8$.

Using result stated in Remark 7, we obtain the following FTS condition for the linear system (1) with time-varying delay.

Corollary 8. *Given a matrix $U > 0$. The linear system (1) with time-varying delay $h(t)$ satisfying Eq. (2) is finite-time stable with respect to $(c_1, c_2, T, h_1, h_2, U)$, $0 \le c_1 < c_2$, if there exist symmetric positive-definite matrices $P, Q_1, Q_2, R_1, R_2, R_3$ and $R_4 \in \mathbb{R}^{n\times n}$ and positive scalars $\alpha, \beta_i, i = 1, 2, \ldots, 8$ such that LMIs and inequalities (11), (18)–(20) hold.*

Next, we use a simpler form of LKF than the one used in Theorem 6 by setting $R_2 = R_4 = 0$. We obtain a new finite-time stability criterion as stated in the following corollary.

Corollary 9. *Given a matrix $U > 0$. The linear system (1) with time-varying delay $h(t)$ satisfying Eq. (2) is finite-time stable with respect to $(c_1, c_2, T, h_1, h_2, U), 0 \le c_1 < c_2$, if there exist symmetric positive-definite matrices P, Q_1, Q_2, R_1, and $R_3 \in \mathbb{R}^{n\times n}$ and positive scalars $\alpha, \beta_i^c, i = 1, 2, 3, 4, 5, 6$ satisfying*

$$\beta_1^c I < \widetilde{P} < \beta_2^c I,\ 0 < \widetilde{Q_1} < \beta_3^c I,\ 0 < \widetilde{Q_2} < \beta_4^c I,\ 0 < \widetilde{R_1} < \beta_5^c I, 0 < \widetilde{R_3} < \beta_6^c I, \tag{21}$$

$$\omega^c = \begin{bmatrix} -\beta_1^c c_2 e^{-\alpha T} & \beta_2^c\sqrt{c_1} & \beta_3^c\sqrt{c_1\gamma_1} & \beta_4^c\sqrt{c_1\gamma_2} & \beta_5^c\sqrt{c_1\gamma_3} & \beta_6^c\sqrt{c_1\gamma_4} \\ * & -\beta_2^c & 0 & 0 & 0 & 0 \\ * & * & -\beta_3^c & 0 & 0 & 0 \\ * & * & * & -\beta_4^c & 0 & 0 \\ * & * & * & * & -\beta_5^c & 0 \\ * & * & * & * & * & -\beta_6^c \end{bmatrix} < 0, \tag{22}$$

and

$$\Psi^c = \begin{bmatrix} \Psi_{11}^c & \Psi_{12}^c & \Psi_{13}^c & 0 & \Psi_{15}^c & 0 & 0 \\ * & \Psi_{22}^c & \Psi_{23}^c & 0 & \Psi_{25}^c & \Psi_{26}^c & 0 \\ * & * & \Psi_{33}^c & \Psi_{34}^c & 0 & \Psi_{36}^c & \Psi_{37}^c \\ * & * & * & \Psi_{44}^c & 0 & 0 & \Psi_{47}^c \\ * & * & * & * & \Psi_{55}^c & 0 & 0 \\ * & * & * & * & * & \Psi_{66}^c & 0 \\ * & * & * & * & * & * & \Psi_{77}^c \end{bmatrix} < 0, \tag{23}$$

where $h_{21} = h_2 - h_1$ *and*

$$\widetilde{P} = U^{-1/2}PU^{-1/2},\quad \widetilde{Q_i} = U^{-1/2}Q_iU^{-1/2},\ i = 1, 2,\quad \widetilde{R_j} = U^{-1/2}R_jU^{-1/2},\ j = 1, 3,$$

$$\Psi_{11}^c = PA_0 + A_0^T P + Q_1 + Q_2 + A_0^T[h_1R_1 + h_{21}R_3]A_0 - \alpha P - \frac{11R_1}{3h_1},\quad \Psi_{12}^c = -\frac{5R_1}{3h_1},$$

$$\Psi_{13}^c = A_0^T[h_1R_1 + h_{21}R_3]A_1 + PA_1,\quad \Psi_{15}^c = \frac{16R_1}{3h_1},\quad \Psi_{22}^c = -e^{\alpha h_1}Q_1 - \frac{8R_1}{3h_1} - \frac{11R_3}{3h_{21}},$$

$$\Psi_{23}^c = -\frac{5R_3}{3h_{21}},\quad \Psi_{25}^c = \frac{13R_1}{3h_1},\quad \Psi_{26}^c = \frac{16R_3}{3h_{21}},\quad \Psi_{33}^c = -\frac{19R_3}{3h_{21}} + A_1^T[h_1R_1 + h_{21}R_3]A_1,$$

$$\Psi_{34}^c = -\frac{5R_3}{3h_{21}},\quad \Psi_{36}^c = \frac{13R_3}{3h_{21}},\quad \Psi_{37}^c = \frac{16R_3}{3h_{21}},\quad \Psi_{44}^c = -e^{\alpha h_2}Q_2 - \frac{8R_3}{3h_{21}},\quad \Psi_{47}^c = \frac{13R_3}{3h_{21}},$$

$$\Psi_{55}^c = -\frac{29R_1}{3h_1},\quad \Psi_{66}^c = -\frac{29R_3}{3h_{21}},\quad \Psi_{77}^c = -\frac{29R_3}{3h_{21}}.$$

Proof. Choose the same LKF as in Theorem 6 with setting $R_2 = R_4 = 0$. Follow the same procedure as in the proof of Theorem 6 and apply Schur's complement as stated in Remark 7, the FTS condition of the linear system (1) with time-varying delay is obtained. □

Remark 10. One can obtain different FTS conditions similar to the one stated in Corollary 9 by setting one of the matrices R_1, R_2, R_4 to be zero matrix or two matrices of R_1, R_2, R_4 to be zero matrices. Note that values of the upper bounds of $x^T(t)Ux(t)$ that guarantee FTS of the linear system with respect to $(c_1, c_2, T, h_1, h_2, U)$ from all cases are very close to one another. The best results (smallest values of the upper bounds of $x^T(t)Ux(t)$) are obtained from Corollary 9 in almost all numerical examples. Thus, we omit some results and only present FTS criteria as in Theorem 6 and Corollary 9 in this article.

Next, we derive a FTS condition of the linear system (1) with time-varying delay by using the Wirtinger-based integral inequality in order to compare the effectiveness with FTS conditions derived by using our new integral inequality (Proposition 4). New FTS condition is stated in Corollary 11.

Corollary 11. *Given a matrix $U > 0$. The linear system (1) with time-varying delay $h(t)$ satisfying Eq. (2) is finite-time stable with respect to $(c_1, c_2, T, h_1, h_2, U), 0 \le c_1 < c_2$, if there exist symmetric positive-definite matrices $P, Q_1, Q_2, R_1, R_2, R_3$ and $R_4 \in \mathbb{R}^{n\times n}$ and positive scalars $\alpha, \beta_i^w, i = 1, 2, \ldots, 8$ satisfying inequalities and LMIs*

(18)–(20) and

$$\Psi^w = \begin{bmatrix} \Psi^w_{11} & \Psi^w_{12} & \Psi^w_{13} & \Psi^w_{14} & \Psi^w_{15} & 0 & 0 & 0 & \Psi^w_{19} \\ * & \Psi^w_{22} & \Psi^w_{23} & \Psi^w_{24} & \Psi^w_{25} & \Psi^w_{26} & \Psi^w_{27} & 0 & 0 \\ * & * & \Psi^w_{33} & \Psi^w_{34} & 0 & \Psi^w_{36} & 0 & \Psi^w_{38} & 0 \\ * & * & * & \Psi^w_{44} & 0 & 0 & \Psi^w_{47} & \Psi^w_{48} & \Psi^w_{49} \\ * & * & * & * & \Psi^w_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Psi^w_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Psi^w_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Psi^w_{88} & 0 \\ * & * & * & * & * & * & * & * & \Psi^w_{99} \end{bmatrix} < 0, \tag{24}$$

where $h_{21} = h_2 - h_1$,

$$\widetilde{P} = U^{-1/2} P U^{-1/2}, \quad \widetilde{Q_i} = U^{-1/2} Q_i U^{-1/2},\ i = 1, 2, \quad \widetilde{R_j} = U^{-1/2} R_j U^{-1/2},\ j = 1, 2, 3, 4,$$

$$\Psi^w_{11} = P A_0 + A_0^T P + Q_1 + Q_2 + A_0^T [h_1 R_1 + h_{21} R_2 + h_{21} R_3 + h_2 R_4] A_0 - \alpha P - \frac{4R_1}{h_1} - \frac{4R_4}{h_2},$$

$$\Psi^w_{12} = -\frac{2R_1}{h_1}, \quad \Psi^w_{13} = A_0^T [h_1 R_1 + h_{21} R_2 + h_{21} R_3 + h_2 R_4] A_1 + P A_1,$$

$$\Psi^w_{14} = -\frac{2R_4}{h_2}, \quad \Psi^w_{15} = \frac{6R_1}{h_1}, \quad \Psi^w_{19} = \frac{6R_4}{h_2}, \quad \Psi^w_{22} = -e^{\alpha h_1} Q_1 - \frac{4R_1}{h_1} - \frac{4R_2}{h_{21}} - \frac{4R_3}{h_{21}},$$

$$\Psi^w_{23} = -\frac{2R_3}{h_{21}}, \quad \Psi^w_{24} = -\frac{2R_2}{h_{21}}, \quad \Psi^w_{25} = \frac{6R_1}{h_1}, \quad \Psi^w_{26} = \frac{6R_3}{h_{21}}, \quad \Psi^w_{27} = \frac{6R_2}{h_{21}},$$

$$\Psi^w_{33} = -\frac{8R_3}{h_{21}} + A_1^T [h_1 R_1 + h_{21} R_2 + h_{21} R_3 + h_2 R_4] A_1, \quad \Psi^w_{34} = -\frac{2R_3}{h_{21}}, \quad \Psi^w_{36} = \frac{6R_3}{h_{21}}, \quad \Psi^w_{38} = \frac{6R_3}{h_{21}},$$

$$\Psi^w_{44} = -e^{\alpha h_2} Q_2 - \frac{4R_2}{h_{21}} - \frac{4R_3}{h_{21}} - \frac{4R_4}{h_2}, \quad \Psi^w_{47} = \frac{6R_2}{h_{21}}, \quad \Psi^w_{48} = \frac{6R_3}{h_{21}}, \quad \Psi^w_{49} = \frac{6R_4}{h_2},$$

$$\Psi^w_{55} = -\frac{12R_1}{h_1} \quad \Psi^w_{66} = -\frac{12R_3}{h_{21}}, \quad \Psi^w_{77} = -\frac{12R_2}{h_{21}}, \quad \Psi^w_{88} = -\frac{12R_3}{h_{21}}, \quad \Psi^w_{99} = -\frac{12R_4}{h_2}.$$

Proof. We follow the same proof as in Theorem 6 except that we apply Lemma 3 to the integrals occurring in Eq. (13). Thus, the FTS condition of the linear system (1) with time-varying delay $h(t)$ is obtained. □

3.2. Finite-time of linear system with constant delay

In this section, we formulate criteria to guarantee the FTS of the linear system with constant delay, $h > 0$, satisfying the following equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h). \tag{25}$$

We derive the FTS conditions of the considered system by first using our new integral inequality (as stated in Theorem 12) and follow by using the Wirtinger-based integral inequality (as stated in Corollary 13).

Theorem 12. *Given a matrix $U > 0$. The linear system (25) with constant time delay h is finite-time stable with respect to (c_1, c_2, T, h, U), $0 \le c_1 < c_2$, if there exist symmetric positive-definite matrices $P, Q, R \in \mathbb{R}^{n\times n}$ and positive scalars $\alpha, \beta_i, i = 9, 10, 11, 12$ satisfying*

$$\beta_9 I < \widetilde{P} < \beta_{10} I, \ 0 < \widetilde{Q} < \beta_{11} I, \ 0 < \widetilde{R} < \beta_{12} I, \tag{26}$$

$$\hat{\omega} = \begin{bmatrix} -\beta_9 c_2 e^{-\alpha T} & \beta_{10}\sqrt{c_1} & \beta_{11}\sqrt{c_1\gamma_6} & \beta_{12}\sqrt{c_1\gamma_7} \\ * & -\beta_{10} & 0 & 0 \\ * & * & -\beta_{11} & 0 \\ * & * & * & -\beta_{12} \end{bmatrix} < 0, \tag{27}$$

and

$$\hat{\Psi} = \begin{bmatrix} \hat{\Psi}_{11} & \hat{\Psi}_{12} & \hat{\Psi}_{13} \\ * & \hat{\Psi}_{22} & \hat{\Psi}_{23} \\ * & * & \hat{\Psi}_{33} \end{bmatrix} < 0, \tag{28}$$

where $\widetilde{P} = U^{-1/2}PU^{-1/2}$, $\widetilde{Q} = U^{-1/2}QU^{-1/2}$, $\widetilde{R} = U^{-1/2}RU^{-1/2}$,

$$\hat{\Psi}_{11} = PA_0 + A_0^T P + Q + hA_0^T RA_0 - \alpha P - \frac{11R}{3h}, \quad \hat{\Psi}_{12} = hA_0^T RA_1 + PA_1 - \frac{5R}{3h}, \quad \hat{\Psi}_{13} = \frac{16R}{3h},$$

$$\hat{\Psi}_{22} = hA_1^T RA_1 - e^{\alpha h}Q - \frac{8R}{3h}, \quad \hat{\Psi}_{23} = \frac{13R}{3h}, \quad \hat{\Psi}_{33} = -\frac{29R}{3h}.$$

Proof. From the linear system (25), we have

$$h\dot{x}^T(t)R\dot{x}(t) = hx^T(t)A_0^T RA_0 x(t) + 2hx^T(t)A_0^T RA_1 x(t-h) + hx^T(t-h)A_1^T RA_1 x(t-h). \tag{29}$$

Choose the LKF in the form $V(x(t)) = \sum_{i=1}^{3} V_i(x(t))$ where

$$V_1(x(t)) = x^T(t)Px(t), \quad V_2(x(t)) = \int_{t-h}^{t} e^{\alpha(t-s)} x^T(s)Qx(s)ds,$$

$$V_3(x(t)) = \int_{-h}^{0}\int_{t+s}^{t} e^{\alpha(t-\theta)}\dot{x}^T(\theta)R\dot{x}(\theta)d\theta ds.$$

We take the derivative of $V(x(t))$ along the solution in Eq. (25) and then apply Eq. (29) to the term $h\dot{x}^T(t)R\dot{x}(t)$. This leads to

$$\begin{aligned}\dot{V}(x(t)) - \alpha V(x(t)) \le{}& x^T(t)\big[PA_0 + A_0^T P + Q + hA_0^T RA_0 - \alpha P\big]x(t) - \int_{t-h}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R\dot{x}(s)ds \\ &+ 2x^T(t)\big[hA_0^T RA_1 + PA_1\big]x(t-h) + x^T(t-h)\big[hA_1^T RA_1 - e^{\alpha h}Q\big]x(t-h).\end{aligned} \tag{30}$$

Define $\hat{\eta}(t) = \big[x^T(t) \quad x^T(t-h) \quad \frac{1}{h}\int_{t-h}^{t} x^T(s)ds\big]^T$. Applying Proposition 4 to the integral term in Eq. (30) along with the fact that $1 \le e^{\alpha(t-s)}, \forall s \in [t-h, t]$, we obtain

$$-\int_{t-h}^{t} e^{\alpha(t-s)}\dot{x}^T(s)R\dot{x}(s)ds \le -\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)ds \le -\frac{\hat{\eta}^T(t)}{6h}\begin{bmatrix} 22R & 10R & -32R \\ * & 16R & -26R \\ * & * & 58R \end{bmatrix}\hat{\eta}(t). \tag{31}$$

Substituting the relation (31) into Eq. (30), we get $\dot{V}(x(t)) - \alpha V(x(t)) \le \hat{\eta}^T(t)\hat{\Psi}\eta(t)$. Since $\hat{\Psi} < 0$ as defined in Eq. (28), this implies $\dot{V}(x(t)) - \alpha V(x(t)) < 0$. With the fact that $I = U^{-1/2}U^{1/2} = U^{1/2}U^{-1/2}$, we multiply this inequality by $e^{-\alpha t}$ then integrate from 0 to t with $t \in [0, T]$ and obtain $e^{-\alpha t}V(x(t)) < V(x(0))$ with

$$\begin{aligned}V(x(0)) &= x^T(0)U^{1/2}\widetilde{P}U^{1/2}x(0) + \int_{-h}^{0} e^{-\alpha s}x^T(s)U^{1/2}\widetilde{Q}U^{1/2}x(s)ds + \int_{-h}^{0}\int_{s}^{0} e^{-\alpha\theta}\dot{x}^T(\theta)U^{1/2}\widetilde{R}U^{1/2}\dot{x}(\theta)d\theta ds \\ &\le c_1\Big[\lambda_{max}(\widetilde{P}) + \lambda_{max}(\widetilde{Q})\int_{-h}^{0} e^{-\alpha s}ds + \lambda_{max}(\widetilde{R_1})\int_{-h}^{0}\int_{s}^{0} e^{-\alpha\theta}d\theta ds\Big] \\ &\le c_1\Big[\lambda_{max}(\widetilde{P}) + \gamma_6\lambda_{max}(\widetilde{Q}) + \gamma_7\lambda_{max}(\widetilde{R})\Big].\end{aligned} \tag{32}$$

Since $V(x(t)) \ge V_1(x(t)) = x^T(t)U^{1/2}\widetilde{P}U^{1/2}x(t) \ge \lambda_{min}(\widetilde{P})x^T(t)Ux(t)$. Thus, Eq. (32) leads to

$$x^T(t)Ux(t) \le \frac{c_1 e^{\alpha T}}{\lambda_{min}(\widetilde{P})}\Big[\lambda_{max}(\widetilde{P}) + \gamma_6\lambda_{max}(\widetilde{Q}) + \gamma_7\lambda_{max}(\widetilde{R})\Big]. \tag{33}$$

Now we define

$$\hat{\Pi} :\equiv c_1\{\beta_{10} + \gamma_6\beta_{11} + \gamma_7\beta_{12}\} - c_2\beta_9 e^{-\alpha T} < 0. \tag{34}$$

Applying Schur's complement lemma, this implies that condition in the relation (33) is equivalent to the inequalities and LMI (26)–(27). Therefore, the FTS condition of the linear system (25) is obtained. □

Next, we derive a FTS condition for the linear system (25) with constant delay using the Wirtinger-based integral inequality to bound the integral term occurring in the derivation. Result is stated in Corollary 13.

Corollary 13. *Given a matrix $U > 0$. The linear system (25) with constant delay h is finite-time stable with respect to (c_1, c_2, T, h, U), $0 \leq c_1 < c_2$, if there exist symmetric positive-definite matrices $P, Q, R \in \mathbb{R}^{n\times n}$ and positive scalars $\alpha, \beta_i, i = 9, 10, 11, 12$ satisfying inequalities and LMIs (26)–(27) and*

$$\hat{\Psi}^w = \begin{bmatrix} \hat{\Psi}^w_{11} & \hat{\Psi}^w_{12} & \hat{\Psi}^w_{13} \\ * & \hat{\Psi}^w_{22} & \hat{\Psi}^w_{23} \\ * & * & \hat{\Psi}^w_{33} \end{bmatrix} < 0, \tag{35}$$

where $\widetilde{P} = U^{-1/2}PU^{-1/2}, \quad \widetilde{Q} = U^{-1/2}QU^{-1/2}, \quad \widetilde{R} = U^{-1/2}RU^{-1/2},$

$$\hat{\Psi}^w_{11} = PA_0 + A_0^T P + Q + hA_0^T RA_0 - \alpha P - \frac{4R}{h}, \quad \hat{\Psi}^w_{12} = hA_0^T RA_1 + PA_1 - \frac{2R}{h},$$
$$\hat{\Psi}^w_{13} = \frac{6R}{h}, \quad \hat{\Psi}^w_{22} = hA_1^T RA_1 - e^{\alpha h}Q - \frac{4R}{h}, \quad \hat{\Psi}^w_{23} = \frac{6R}{h}, \quad \hat{\Psi}^w_{33} = -\frac{12R}{h}.$$

Proof. We choose the same LKF as in Theorem 12, then follow the same proof but bounding the integral term occurring in the relation (30) by the Wirtinger-based integral inequality (Lemma 3). Thus, the proof is complete. □

4. Numerical examples

In this section, two numerical examples are given to show the effectiveness by comparing the smallest eligible c_2 guaranteeing FTS of linear system among the proposed criteria and some existing works. To find the smallest eligible c_2 guaranteeing FTS of linear system, we solve required inequalities and LMIs stated in the proposed criteria using MATLAB control toolbox. In the first example, we compare the smallest c_2 guaranteeing FTS of linear system with constant delay. In the second example, FTS of linear system with time-varying delay is investigated.

Example 1. Consider the linear system with constant delay as in Eq. (25) with

$$A_0 = \begin{bmatrix} -1.7 & 1.7 & 0 \\ 1.3 & -1 & 0.7 \\ 0.7 & 1 & -0.6 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1.5 & -1.7 & 0.1 \\ -1.3 & 1 & -0.3 \\ -0.7 & 1 & 0.6 \end{bmatrix}. \tag{36}$$

In this example, FTS of the given linear system with constant delay is considered in two cases with two different initial conditions; $\phi(\theta) = [0.4, \ 0.2, \ 0.4]^T$ and $\phi(\theta) = [0.7, \ 0, \ 0]^T$.

Case 1: FTS of the linear system in Eq. (25) with A_0, A_1 defined in Eq. (36) with respect to $(c_1 = 0.36, c_2, T, h = 0.2, U = I)$ for various c_2 and T.

With a constant delay $h = 0.2$ and an initial condition $\phi(\theta) = [0.4, \ 0.2, \ 0.4]^T$, we plot the solution of the considered linear system in Fig. 1. One can notice the values of $x^T(t)Ux(t) > 100$ for some $t \in [0, 4]$ (see Fig. 1 right). Indeed, this considered system is not asymptotically stable. Because $x^T(0)Ux(0) = \|\phi\|^2 = 0.36$, thus, we let $c_1 = 0.36$ for further FTS investigation.

By solving inequalities and LMIs (26)–(28) (for Theorem 12) and (26), (27), (35) (for Corollary 13) and comparing with results using Theorem 1 in [17], results are listed in Table 1. One can see that Theorem 12 guarantees FTS of the linear system for fixed c_2 and T in all cases. For the same T, Corollary 13 guarantees FTS with larger values of c_2 but Theorem 1 in [17] does not guarantee FTS in all cases. For example, with given $(c_1, c_2, T, h, U) = (0.36, 32, 2, 0.2, I)$, Theorem 12 guarantees FTS and yields the upper bound of $x^T(t)Ux(t) = 31.9103 < 32 = c_2$; while Corollary 13 guarantees FTS with higher value of $c_2 = 44$.

Case 2: FTS of the linear system in Eq. (25) with A_0, A_1 defined in Eq. (36) with respect to $(c_1 = 0.55, c_2, T = 2, h, U = I)$ for two constant delays $h = 0.2$ and $h = 0.5$ and various c_2.

With the initial condition $\phi(\theta) = [0.7, \ 0, \ 0]^T$, we have $x^T(0)Ux(0) = \|\phi\|^2 = 0.49$. Thus, we choose $c_1 = 0.55$. First, we compare the smallest eligible values of c_2 guaranteeing FTS of the linear system among our proposed criteria (Theorem 12, Corollary 13) and criteria from other works. Results are listed in Table 2. One can see that Theorem 12 guarantees FTS of the linear system with smallest value of $c_2 = 48$ for both $h = 0.2$ and $h = 0.5$; while

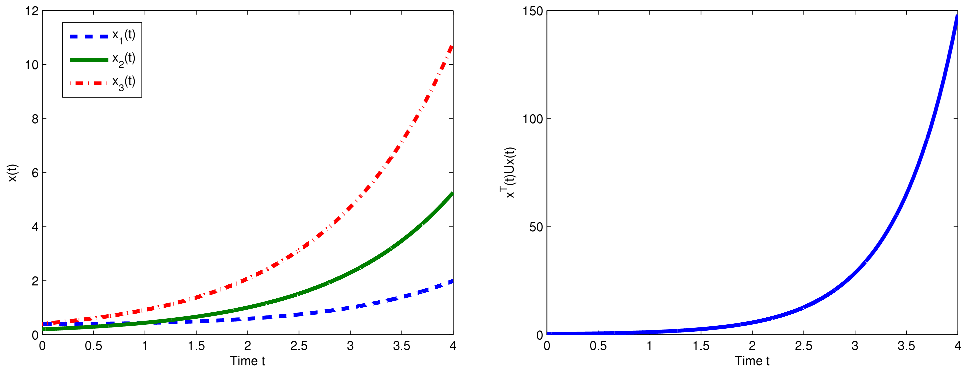

Fig. 1. Time history of state variables (left) and $x^T(t)Ux(t)$ (right) with $h = 0.2, \phi(t) = [0.4\ 0.2\ 0.4]^T$ for $t \in [0, 4]$.

Table 1
Upper bounds of computed $x^T(t)Ux(t), t \in [0, T]$ along with c_2 guaranteeing FTS of linear system with respect to $(0.36, c_2, T, 0.2, I)$. Numbers in parentheses are values of α. NF means 'not FTS'.

	$T = 1$		$T = 2$		$T = 3$		$T = 4$	
	$c_2 = 4$	$c_2 = 5.8$	$c_2 = 32$	$c_2 = 44$	$c_2 = 200$	$c_2 = 270$	$c_2 = 1164$	$c_2 = 1534$
Theorem 12 (New inequality)	3.9986 (2.123)	5.5983 (2.025)	31.9103 (1.962)	42.3108 (1.945)	199.8691 (1.809)	261.5847 (1.760)	1163.96 (1.744)	1492 (1.733)
Corollary 13 (Wirtinger Inequality)	NF	4.3870 (2.224)	NF	35.0414 (1.881)	NF	220.0391 (1.7810)	NF	1280.78 (1.727)
Theorem 1 (Lin et al. [17])	NF	NF	NF	NF	NF	NF	NF	NF
$max_{[0,T]} x^T(t)Ux(t)$	1.227		5.677		28.662		148.64	

Table 2
Upper bounds of computed $x^T(t)Ux(t), t \in [0, 2]$ along with c_2 guaranteeing FTS of linear system with $c_1 = 0.55, T = 2, U = I$ for $h = 0.2, 0.5$. Numbers in parentheses are values of α. NF means 'not FTS'.

h	$c_2 = 48$		$c_2 = 68$	$c_2 = 60$	$c_2 = 100$	
	0.2	0.5	0.2	0.5	0.2	0.5
Lin et al. [17]	NF	NF	NF	NF	NF	NF
Rojsiraphisal & Puangmalai [23]	NF	NF	NF	NF	79.18	NF
Stojanovic et al. [26]	NF	NF	NF	NF	95.59	NF
Theorem 12 (New inequality)	47.4679 (1.9420)	47.1496 (1.818)	65.3652 (1.913)	58.9704 (1.789)	89.3516 (1.830)	91.3206 (1.737)
Corollary 13 (Wirtinger inequality)	NF	NF	53.154 (1.907)	46.2822 (1.825)	57.9501 (1.880)	69.5813 (1.711)

some conditions guarantee FTS of the linear system with larger values of c_2 except Theorem 1 in [17] that does not guarantee FTS. Note that, for $h = 0.2$, Theorem 12 provides the value of $x^T(t)Ux(t) = 47.4679 < 48 = c_2$; while Corollary 13, conditions in [23] and [26] require $c_2 = 68, 100, 100$ to guarantee FTS, respectively. Next, consider delay $h = 0.5$, Corollary 13 guarantees FTS with $c_2 = 60$ but criteria from [23,26] do not guarantee FTS with the delay $h = 0.5$. Thus, Theorem 12 and Corollary 13 are less conservative than previous works.

Remark 14. From Example 1, we can see that FTS condition for linear system with constant delay proposed in Theorem 12 derived using the new proposed integral inequality (Proposition 4) yields less conservative than condition proposed in Corollary 13 derived using the Wirtinger-based integral inequality.

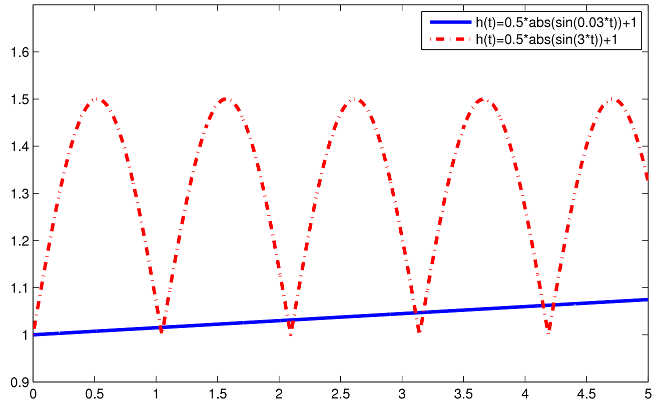

Fig. 2. Continuous delay function for $t \in [0, 5]$.

Table 3
Upper bounds of computed $x^T(t)Ux(t), t \in [0, T]$ along with c_2 guaranteeing FTS of linear system with $c_1 = 0.18, h_1 = 1, h_2 = 1.5, U = I$ and $T = 1, 2, 3, 4, 5$. Numbers in parentheses are values of α.

T	1	2	3	4	5
c_2	0.41	0.43	0.45	0.47	0.49
Corollary 8 (New inequality & $R_1 - R_4$)	0.4091 (0.071)	0.4296 (0.045)	0.4496 (0.041)	0.4696 (0.038)	0.4895 (0.037)
Corollary 9 (New inequality & R_1, R_3)	0.4086 (0.079)	0.4290 (0.051)	0.4491 (0.043)	0.4690 (0.043)	0.4890 (0.037)
Corollary 11 (Wirtinger inequality & R_1–R_4)	0.4062 (0.003)	0.4253 (0.009)	0.4450 (0.001)	0.4640 (0.001)	0.4828 (0.001)
Theorem 1 Lin et al. [17] Stojanovic [25] Zhang et al. [31]	NF 0.33 759.5	NF 0.33 1.21^6	NF 0.33 1.79^9	NF 0.33 2.58^{12}	NF 0.33 3.72^{15}

Example 2. Consider the linear system with time-varying delay as in Eq. (1) where

$$A_0 = \begin{bmatrix} -0.2 & 2 \\ -1 & -0.2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}, \tag{37}$$

and time-varying delay satisfying Eq. (2) with an initial condition $\phi(\theta) = [0.1\theta + 0.2,\ -0.1\theta - 0.2]^T, \theta \in [-h_2, 0]$.

Case 1: FTS of the linear system in Eq. (1) with A_0, A_1 defined in Eq. (37) and continuous time-varying delay $h(t) = 0.5|sin(0.03t)| + 1$ for $t \in [0, T], T = 1, 2, 3, 4, 5$.

Notice that this considered delay function is continuous and differentiable for $0 \leq t \leq 5$ (see Fig. 2). With this delay function, we have $0 < 1 \leq h(t) \leq 1.5$. From the initial condition, we have $x^T(0)Ux(0) = \|\phi(\theta)\|^2 = sup_{\theta\in[-h_2,0]}\{0.02(\theta^2 + 4\theta + 4), 0.02\} = 0.08$. Here, we choose $h_1 = 1, h_2 = 1.5$, and $c_1 = 0.18$ for investigating the FTS with respect to $(c_1 = 0.18, c_2, T, h_1 = 1, h_2 = 1.5, U = I)$ of this system.

Comparing the criteria proposed in Section 3.1 with existing conditions, results are listed in Table 3. One can see that, for given c_2 and T, values of the upper bounds of $x^T(t)Ux(t)$ from Corollary 11 which is derived by using the Wirtinger-based integral inequality yield smaller values than the other conditions except results from condition in [25]. Notice that Corollary 8 required to determine two more variables than Corollary 9 but the former condition yields larger values of $x^T(t)Ux(t)$ than those in the latter condition. It is worth noting that Theorem 6 does not guarantee FTS for all cases (results are not shown).

We further investigate FTS of this system using Corollary 11 with smaller value of $c_2 = 0.36$ for $T = 1, 2, 3, 4, 5$. With $\alpha = 0.001$, Corollary 11 guarantees FTS with the upper bounds of $x^T(t)Ux(t) = 0.3596 < c_2$ for all given T. The value of $c_2 = 0.36$ guaranteeing FTS of the system is slightly larger than $c_2 = 0.33$ given by the condition

Table 4
Upper bounds of $x^T(t)Ux(t), t \in [0, T]$ along with c_2 guaranteeing FTS of linear system with $c_1 = 0.18, U = I, T = 1, 2$ and various intervals of $[h_1, h_2]$. Numbers in parentheses are values of α.

	$[h_1, h_2]$ c_2	$[0.1, 0.6]$ 0.30	$[0.2, 0.7]$ 0.31	$[0.3, 0.8]$ 0.32	$[0.4, 0.9]$ 0.33	$[0.5, 1]$ 0.4
$T = 1$	Corollary 8 (New inequality & $R_1 - R_4$)	0.2997 (0.001)	0.3095 (0.004)	0.3196 (0.001)	0.3297 (0.001)	0.3395 (0.001)
	Corollary 9 (New inequality & R_1, R_3)	0.2994 (0.002)	0.3093 (0.001)	0.3194 (0.001)	0.3293 (0.001)	0.3392 (0.001)
	Corollary 11 (Wirtinger inequality & $R_1 - R_4$)	0.2977 (0.003)	0.3074 (0.003)	0.3178 (0.001)	0.3281 (0.001)	0.3377 (0.001)
$T = 2$	Corollary 8 (New inequality & $R_1 - R_4$)	0.2996 (0.002)	0.3095 (0.001)	0.3196 (0.001)	0.3296 (0.003)	0.3397 (0.001)
	Corollary 9 (New inequality & R_1, R_3)	0.2995 (0.003)	0.3095 (0.001)	0.3192 (0.002)	0.3293 (0.001)	0.3393 (0.001)
	Corollary 11 (Wirtinger inequality & $R_1 - R_4$)	0.2977 (0.001)	0.3074 (0.001)	0.3178 (0.001)	0.3282 (0.002)	0.3378 (0.001)

in Stojanovic [25] but smaller than results provided in Zhang et al. [31]. However, Theorem 1 in [17] does not guarantee FTS in all cases.

Case 2: FTS of the linear system in Eq. (1) with A_0, A_1 defined in Eq. (37), various interval of continuous time-varying delays $h(t) = 0.5|sin(0.03t)| + 0.1k, k = 1, 2, 3, 4, 5,\ t \in [0, T]$ with $T = 1, 2$ and $\phi(\theta) = [0.1\theta + 0.2,\ -0.1\theta - 0.2]^T, \theta \in [-h_2, 0]$.

Here, we investigate the effect of changing lower bounds of delay functions with keeping the width of each delay function to be 0.5. Results of FTS of this system with respect to $(c_1 = 0.18, c_2, T, h_1, h_2, U = I), T = 1, 2$ are shown in Table 4. For given $c_2 = 0.3, 0.31, 0.32, 0.33$ and 0.4, we observe that the upper bounds of $x^T(t)Ux(t)$ obtained from Corollary 11 are smaller than those obtained from Corollaries 8–9. Note that the upper bounds of $x^T(t)Ux(t)$ from Corollaries 8 and 9 are slightly different in the fourth digit with smaller values of upper bounds $x^T(t)Ux(t)$ in Corollary 9. In addition, we observe that all criteria require larger values of c_2 to guarantee FTS when k increases.

Case 3: FTS of the linear system in Eq. (1) with A_0, A_1 defined in Eq. (37), various intervals of continuous time-varying delays $h(t) = 0.5k|sin(0.03t)| + 1,\ k = 2, 3, 4, 5, t \in [0, T]$ with $T = 1$ and $\phi(\theta) = [0.1\theta + 0.2,\ -0.1\theta - 0.2]^T, \theta \in [-h_2, 0]$.

In this case, we investigate the effect of changing the widths of interval time-varying delays when $h_1 = 1$ is fixed. Results of FTS of this system with respect to $(c_1 = 0.18, c_2, T = 1, h_1 = 1, h_2, U = I)$ are shown in Table 5. The computed values of the upper bounds of $x^T(t)Ux(t)$ obtained from Corollary 11 are smaller than those obtained from Corollaries 8–9. Similar to case 2, the upper bounds of $x^T(t)Ux(t)$ from Corollaries 8 and 9 are only different in the fourth digit with smaller values of $x^T(t)Ux(t)$ found in the latter case. Also, we notice that all criteria require larger values of c_2 to guarantee FTS of the system when k increases.

Remark 15. Based on numerical example of FTS of linear system with time-varying delay, we observe the following. (i) FTS condition of linear system with time-varying delay in Corollary 11 derived by using the Wirtinger-based integral inequality is less conservative than the conditions derived by using the proposed integral inequality in Proposition 4. (ii) FTS condition derived by using Proposition 4 as stated in Theorem 6 is not FTS in all cases because the condition (12) does not satisfy. However, Corollary 8 which replaces condition (12) by certain inequalities and LMI guarantees FTS in many cases. (iii) FTS condition in Corollary 9, which is derived using a modified LKF in Corollary 8 by setting $R_2 = R_3 = 0$, provides smaller values of the upper bounds of $x^T(t)Ux(t)$ and requires less numbers of variables to be determined among the FTS criteria derived by using the integral inequality in Proposition 4. So condition in Corollary 9 is the best choice among the conditions derived using the proposed integral inequality. Note that some results are not shown here.

Table 5
Upper bounds of $x^T(t)Ux(t), t \in [0, 1]$ along with c_2 guaranteeing FTS of linear system with $c_1 = 0.18, U = I$ and various intervals $[h_1, h_2]$. Numbers in parentheses are values of α.

$[h_1, h_2]$	$[1, 2]$	$[1, 2.5]$	$[1, 3]$	$[1, 3.5]$
c_2	0.51	0.66	0.92	1.46
Corollaries 8 (New inequality & $R_1 - R_4$)	0.5097 (0.225)	0.6599 (0.384)	0.9198 (0.566)	1.4597 (0.748)
Corollary 9 (New inequality & R_1, R_3)	0.5093 (0.238)	0.6597 (0.394)	0.9195 (0.570)	1.4596 (0.743)
Corollary 11 (Wirtinger inequality & $R_1 - R_4$)	0.5076 (0.178)	0.6548 (0.320)	0.9075 (0.525)	1.4368 (0.737)

Remark 16. We also modified FTS condition derived using the Wirtinger-based integral inequality, as stated in Corollary 11, by setting some of the matrices R_1, R_2, R_4 to be zero(es). Results are not shown here. We observe that condition as stated in Corollary 11 is the best condition among these conditions for guaranteeing FTS of linear system with time-varying delay.

Remark 17. Reinvestigating the FTS in Example 2 with continuous time-varying delay satisfying $h(t) = 0.5|sin(3t)| + 1, t \in [0, 5]$. Same results as listed in Table 3 are obtained when Corollaries 8–9 and Corollary 11 are applied. However, condition from Stojanovic [25] does not guarantee FTS for the same system because this condition requires delay function to be continuous and differentiable. But the considered delay function, in this case, is continuous but not differentiable for $0 \leq t \leq 5$ (as seen in Fig. 2).

5. Conclusions

In this research, a new integral inequality has been proposed. The new inequality has been used to derive the FTS conditions of linear system with time-varying and constant delays. The FTS criteria derived by using the new integral inequality have been compared with the FTS conditions formulated by using the Wirtinger-based integral inequality. Numerical results show that the FTS criteria derived by using the new integral inequality are less conservative than the ones derived by using the Wirtinger-based integral inequality when linear systems with constant delays are examined. In contrast, the FTS criteria derived by using the Wirtinger-based integral inequality are less conservative than the ones derived by using the new integral inequality when linear systems with time-varying delays are considered.

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