Complex Valued Rectangular B-Metric Spaces and Fixed Point Theorems

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Abstract

In this paper, we study the existence of fixed point for self mappings under generalized kannan mappings type concept in rectangular b-metric spaces. Our result extend and generalize the result derived by Ege [21] and many others.

Keywords: complex valued rectangular b-metric spaces / fixed point theorems / kannan mappings.

1. Introduction

The Banach fixed point theorem still seems to be the most important result in metric fixed point theory. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations and other related areas. Metric fixed point theory the first important and significant result was proved by Banach in 1922 for contraction mapping in complete metric space was introduced by Frechet [1]. In 2000, Branciari [2] introduced a notion of rectangular metric space and proved an analogue of the Banach contraction principle in this space, then various fixed point theorems were given for different contractions on rectangular metric spaces (see [3] – [11]). In 2011, Azam and et al. [12] introduced the complex valued b-metric spaces and proved common fixed point results for mappings.

On the other hand, in 1989, Bakhtin [13] introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces ([14] – [20] and the references therein). In 2015, Ege [21] introduced complex valued rectangular b-metric spaces and proved an analogue of Banach contraction principle. Moreover, author also prove a different contraction principle with a new condition and a fixed point theorem in this space.

Research objectives

1. We will study the existence of fixed point for self mappings under generalized kannan mappings type concept in complex valued rectangular b-metric spaces.

2. We will give examples as a satisfying the theorems in complex valued rectangular b-metric spaces.

2. Preliminaries

In this section, we review the basic knowledge to prove our main results.

Definition 2.1. [4] Let X be a nonempty set and $d: X \times X \to [0,\infty)$ be a satisfies:

(bM1) $d(x,y)=0$ if and only if $x=y$ for all $x,y \in X$;

(bM2) $d(x,y)= d(y,x)$ for all $x,y \in X$;

(bM3) there exists a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$. Then d is a $b-$metric on X and (X,d) is called a $b-$*metric space (in short bMS)* with coefficient s.

Definition 2.2. [5] Let X be a nonempty set and the mapping $d: X \times X \to [0,\infty)$ satisfies:

(RM1) $d(x,y)=0$ if and only if $x=y$ for all $x,y \in X$;

(RM2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(RM3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all $x,y \in X$ and all distinct points $u,v \in X \setminus \{x,y\}$. Then d is called a rectangular metric on X and (X,d) is called a *rectangular metric space (in short RMS).*

Definition 2.3. [22] Let X be a nonempty set and the mapping $d: X \times X \to [0,\infty)$ satisfies:

(RbM1) $d(x,y)=0$ if and only if $x=y$ for all $x,y \in X$;

(RbM2) $d(x,y)= d(y,x)$ for all $x,y \in X$;

(RbM3) there exists a real number $s \geq 1$such that

$d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$for all $x,y \in X$ and all distinct points $u,v \in X \setminus \{x,y\}$.

Then d is called a rectangular $b-$metric on X and (X,d) is called a *rectangular* $b-$*metric space (in short RbMS) with coefficient* s.

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse of the above implication is not necessarily true.

Example 2.4. [22] Let $X = \mathbb{N}$, define $d: X \times X \to X$ by

$$d(x,y) = \begin{cases} 0, & if\ x = y; \\ 4\alpha, & if\ x,y \in \{1,2\}\ and\ x \neq y; \\ \alpha\ , & if\ x\ or\ y \notin \{1,2\}\ and\ x \neq y, \end{cases}$$

Where $\alpha > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (X,d) is not a rectangular metric space, as

$d(1,2) = 4\alpha > 3\alpha = d(1,3) + d(3,4) + d(4,2).$

Example 2.5. [22] Let $X = \mathbb{N}$, define $d: X \times X \to X$ such that $d(x,y) = d(y,x)$ for all $x,y \in X$ and

$$d(x,y) = \begin{cases} 0, & if\ x = y; \\ 10\alpha, & if\ x = 1,\ y = 2; \\ \alpha\ , & if\ x \in \{1,2\}\ and\ y \in \{3\}; \\ 2\alpha\ , & if\ x \in \{1,2,3\}\ and\ y \in \{4\}; \\ 3\alpha\ , & if\ x\ or\ y \notin \{1,2,3,4\}\ and\ x \neq y, \end{cases}$$

Where $\alpha > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $s = 2 > 1$, but (X,d) is not a rectangular metric space, as

$d(1,2) = 10\alpha > 5\alpha = d(1,3) + d(3,4) + d(4,2).$

The complex metric space was initiated by Azam et al. [5]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Particularly, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (C_2),(C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:

(1) If $a,b \in \mathbb{R}$ with $a < b$, then $az \prec bz$ for all $z \in \mathbb{C}$.

(2) If $0 \preceq z_1 \lnsim z_2$, then $|z_1| < |z_2|$.

(3) If $z_1 \preceq z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 2.6.[21] Let X be a nonempty set. Suppose that a mapping $d : X \times X \to \mathbb{C}$ satisfies:

(CRb1) $d(x,y) = 0$ if and only if $x = y$ for all $x,y \in X$;

(CRb2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(CRb3) there exists a real number $s \geq 1$ such that

$d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ for all $x, y \in X$ and all distinct points $u,v \in X \setminus \{x,y\}$.

Then d is called a complex valued rectangular $b-$metric on X and (X,d) is called a complex valued rectangular $b-$metric space.

Example 2.7. [21] Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \mathbb{Z}^+$ and $d : X \times X \to \mathbb{C}$ be defined as follows: $d(x,y) = d(y,x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & if\ x = y; \\ 2t, & if\ x, y \in A; \\ \frac{t}{2n}, & if\ x \in A\ and\ y \in \{2,3\}; \\ t, & otherwise, \end{cases}$$

where $t > 0$ is a constant. Then (X,d) is a rectangular $b-$metric space with coefficient $s = 2 > 1$.

Definition 2.8. [21] Let (X,d) be a complex valued rectangular $b-$metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(a) The sequence $\{x_n\}$ is said to be complex valued convergent in (X,d) and converges to x if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and is denoted by $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is called complex valued Cauchy sequence in (X,d) if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(c) (X,d) is said to be a complex valued complete rectangular $b-$metric space if every complex valued Cauchy sequence in X converges to some $x \in X$.

Since the following two lemmas are the analogues of the lemmas in [5], we state these for complex valued rectangular $b-$metric spaces without their proofs.

Lemma 2.9. [21] Let (X,d) be a complex valued rectangular $b-$metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to X if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.10. [21] Let (X,d) be a complex valued rectangular $b-$metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Theorem 1. [21] Let (X,d) be a complex valued complete rectangular $b-$metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:

$$d(Tx, Ty) \leq \alpha d(x,y)$$

for all $x, y \in X$, where $\alpha \in \left[0, \frac{1}{s}\right]$. Then T has a unique fixed point.

3. Results

Now we prove our main results.

Theorem 2. Let (X, d) be a complete complex valued rectangular $b-$metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:

$$d(Tx, Ty) \le \lambda\left[d(x, Tx) + d(y, Ty)\right] \quad (3.1)$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{2}{s+1}\right]$. Then T has a unique fixed point.

Proof Let $z_0 \in X$ and each $n \in \mathbb{N}$, we define $z_n = Tz_{n-1};\ for\ all\ n \in \mathbb{N}$.

We consider,

$$\begin{aligned} d(z_1, z_2) &= d(Tz_0, Tz_1) \\ &\le \lambda[d(z_0, Tz_0) + d(z_1, Tz_1)] \\ &\le \lambda[d(z_0, z_1) + d(z_1, z_2)], \quad (3.2) \end{aligned}$$

$$\Rightarrow \quad d(z_1, z_2) \le \lambda d(z_0, z_1) + \lambda d(z_1, z_2)$$

$$\Rightarrow d(z_1, z_2) - \lambda d(z_1, z_2) \le \lambda d(z_0, z_1)$$

$$\Rightarrow \quad (1-\lambda) d(z_1, z_2) \le \lambda d(z_0, z_1)$$

$$\Rightarrow \quad d(z_1, z_2) \le \frac{\lambda d(z_0, z_1)}{(1-\lambda)},$$

$$d(z_1, z_2) \le \left(\frac{\lambda}{1-\lambda}\right) e_0;\ where\ e_0 = d(z_0, z_1). \quad (3.3)$$

Thus, $d(z_1, z_2) \le \left(\frac{\lambda}{1-\lambda}\right) e_0,\ for\ some\ \lambda \in \left[0, \frac{2}{s+1}\right], where\ e_0 = d(z_0, z_1).$

$$\begin{aligned} d(z_2, z_3) &= d(Tz_1, Tz_2) \\ &\le \lambda[d(z_1, Tz_1) + d(z_2, Tz_2)] \\ &\le \lambda[d(z_1, z_2) + d(z_1, z_3)], \quad (3.4) \end{aligned}$$

$$\Rightarrow \quad d(z_2, z_3) \le \lambda d(z_1, z_2) + \lambda d(z_2, z_3)$$

$$\Rightarrow d(z_2, z_3) - \lambda d(z_2, z_3) \le \lambda d(z_1, z_2)$$

$$\Rightarrow \quad (1-\lambda) d(z_2, z_3) \le \lambda d(z_1, z_2)$$

$$\Rightarrow \quad d(z_2, z_3) \le \frac{\lambda d(z_1, z_2)}{(1-\lambda)},$$

$$d(z_2, z_3) \le \left(\frac{\lambda}{1-\lambda}\right) e_1;\ where\ e_1 = d(z_1, z_2). \quad (3.5)$$

Thus, $d(z_2,z_3)\le\left(\frac{\lambda}{1-\lambda}\right)e_1,\ for\ some\ \lambda\in\left[0,\frac{2}{s+1}\right], where\ e_1=d(z_1,z_2).$

Using (3.3) and (3.5) we have

$$d(z_2,z_3)\le\left(\frac{\lambda}{1-\lambda}\right)d(z_1,z_2),$$

$$\le\left(\frac{\lambda}{1-\lambda}\right)\left[\left(\frac{\lambda}{1-\lambda}\right)d(z_0,z_1)\right]$$

$$d(z_2,z_3)=\left(\frac{\lambda}{1-\lambda}\right)^2 e_0,\ for\ some\ \lambda\in\left[0,\frac{2}{s+1}\right];\ where\ e_0=d(z_0,z_1). \qquad (3.6)$$

In general, we obtain that

$$d(z_n,z_{n+1})\le\left(\frac{\lambda}{1-\lambda}\right)^n e_0,\ for\ some\ \lambda\in\left[0,\frac{2}{s+1}\right], where\ e_0=d(z_0,z_1). \qquad (3.7)$$

Let $e_n=d(z_n,z_{n+1}),\forall n\in\mathbb{N}$

$$\begin{aligned}d(z_n,z_{n+2}) &= d(Tz_{n-1},Tz_{n+1})\\ &\le\lambda\left[d(z_{n-1},Tz_{n-1})+d(z_{n+1},Tz_{n+1})\right]\\ &\le\lambda\left[d(z_{n-1},z_n)+d(z_{n+1},z_{n+2})\right]\\ &=\lambda d(z_{n-1},z_n)+\lambda d(z_{n+1},z_{n+2})\\ &\le\lambda\left(\frac{\lambda}{1-\lambda}\right)^{n-1}e_0+\lambda\left(\frac{\lambda}{1-\lambda}\right)^{n+1}e_0,\\ &\le\left(\frac{\lambda}{1-\lambda}\right)^{n-1}\left[\lambda e_0+\lambda\left(\frac{\lambda}{1-\lambda}\right)^2 e_0\right],\end{aligned}$$

$$d(z_n,z_{n+2})\le\left(\frac{\lambda}{1-\lambda}\right)^{n-1}e^*_0\quad where\ e^*_0=\left[\lambda e_0+\lambda\left(\frac{\lambda}{1-\lambda}\right)^2 e_0\right] for\ some\ \lambda\in\left[0,\frac{2}{s+1}\right] \qquad (3.8)$$

$$\begin{aligned}d(z_n,z_{n+2m+1}) &\le s\left[d(z_n,z_{n+1})+d(z_{n+1},z_{n+2})+d(z_{n+2},z_{n+2m+1})\right]\\ &\le s\left[d(z_n,z_{n+1})+d(z_{n+1},z_{n+2})\right.\\ &\quad\left.+s\left[d(z_{n+2},z_{n+3})+d(z_{n+3},z_{n+4})+d(z_{n+4},z_{n+2m+1})\right]\right]\\ &\le s\left[(e_n+e_{n+1})+s\left[(e_{n+2}+e_{n+3})+s\left[(e_{n+4}+e_{n+5})+\ldots\right.\right.\right.\\ &\quad+s\left[(e_{n+2m-2}+e_{n+2m-1}+e_{n+2m})\right.\\ &\le se_n+se_{n+1}+s^2e_{n+2}+s^2e_{n+3}+s^3e_{n+4}+s^3e_{n+5}+\ldots\\ &\quad+s^me_{n+2m-2}+s^me_{n+2m-1}+s^me_{n+2m}\\ &\le s\left[\left(\frac{\lambda}{1-\lambda}\right)^n e_0+\left(\frac{\lambda}{1-\lambda}\right)^{n+1}e_0\right]+s^2\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+2}e_0+\left(\frac{\lambda}{1-\lambda}\right)^{n+3}e_0\right]\\ &\quad+s^3\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+4}e_0+\left(\frac{\lambda}{1-\lambda}\right)^{n+5}e_0\right]+\ldots\\ &\quad+s^m\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-2}e_0+\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-1}e_0+\left(\frac{\lambda}{1-\lambda}\right)^{n+2m}e_0\right]\end{aligned}$$

$$\le \left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+2} e_0 + s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+4} e_0 + \ldots + s^m\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-2} e_0 + s^m\left(\frac{\lambda}{1-\lambda}\right)^{n+2m} e_0 \right]$$

$$+\left[s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+3} e_0 + s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+5} e_0 + \ldots + s^m\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-1} e_0 \right]$$

$$\le s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 \left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-2} + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m}\right]$$

$$+ s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 \left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-2}\right]$$

$$\le s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 \left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right] + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 \left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right]$$

$$\le \left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right]\left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 \right]$$

$$= \frac{1}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}} \left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 \right]$$

$$= \frac{s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 \left(1 + \left(\frac{\lambda}{1-\lambda}\right)\right)}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}},$$

$$d\left(z_n, z_{n+2m+1}\right) \le \frac{s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 \left(1 + \left(\frac{\lambda}{1-\lambda}\right)\right)}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}}. \qquad (3.9)$$

$$\begin{aligned}
d\left(z_n, z_{n+2m}\right) &\le s\left[d\left(z_n, z_{n+1}\right) + d\left(z_{n+1}, z_{n+2}\right) + d\left(z_{n+2}, z_{n+2m}\right)\right] \\
&\le s\left[d\left(z_n, z_{n+1}\right) + d\left(z_{n+1}, z_{n+2}\right)\right. \\
&\quad + s\left[d\left(z_{n+2}, z_{n+3}\right) + d\left(z_{n+3}, z_{n+4}\right) + d\left(z_{n+4}, z_{n+2m}\right)\right] \\
&\le s\left[\left(e_n + e_{n+1}\right) + s\left[\left(e_{n+2} + e_{n+3}\right) + s\left[\left(e_{n+4} + e_{n+5}\right) + \ldots\right.\right.\right. \\
&\quad + s^{m-1}\left[\left(e_{2m-4} + e_{2m-3}\right) + s^{m-1} d\left(z_{n+2m-2}, z_{n+2m}\right)\right. \\
&\le se_n + se_{n+1} + s^2 e_{n+2} + s^2 e_{n+3} + s^3 e_{n+4} + s^3 e_{n+5} + \ldots \\
&\quad + s^{m-1} e_{2m-4} + s^{m-1} e_{2m-3} + s^{m-1} d\left(z_{n+2m-2}, z_{n+2m}\right)
\end{aligned}$$

$$
\begin{aligned}
d(z_n, z_{n+2m}) &\le s\left[\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right] + s^2\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+2} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+3} e_0\right] \\
&\quad + s^3\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+4} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+5} e_0\right] + \ldots + s^{m-1}\left[\left(\frac{\lambda}{1-\lambda}\right)^{2m-4} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{2m-3}\right] \\
&\quad + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^{*} \\
&\le \left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+2} e_0 + s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+4} e_0 + \ldots + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-4} e_0\right] + \\
&\quad + \left[s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+3} e_0 + s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+5} e_0 + \ldots + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-3} e_0\right] \\
&\quad + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^{*}
\end{aligned}
$$

$$
\begin{aligned}
&\le s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots + s^{m-2}\left(\frac{\lambda}{1-\lambda}\right)^{2m-n-4}\right] \\
&\quad + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots + s^{m-2}\left(\frac{\lambda}{1-\lambda}\right)^{2m-n-4}\right] \\
&\quad + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^{*} \\
&\le s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right] + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right] \\
&\quad + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^{*} \\
&\le \left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right]\left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right] + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-2} e_0^{*} \\
&= \frac{1}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}}\left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right] + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^{*} \\
&= \frac{s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left(1 + \left(\frac{\lambda}{1-\lambda}\right)\right)}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}} + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^{*},
\end{aligned}
$$

$$
d(z_n, z_{n+2m}) \le \frac{s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left(1 + \left(\frac{\lambda}{1-\lambda}\right)\right)}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}} + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^{*}. \tag{3.10}
$$

It follows from (3.9) and (3.10) that

$$\lim_{n\to\infty} d\left(z_n, z_{n+p}\right) = 0 \text{ for all } p > 0. \qquad (3.11)$$

Thus $\{z_n\}$ is a Cauchy sequence in X. By completeness of (X,d) there exists $u \in X$ such that

$$\lim_{n\to\infty} z_n = u. \qquad (3.12)$$

We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,Tu) &\le s\left[d(u,z_n) + d(z_n,z_{n+1}) + d(z_{n+1},Tu)\right] \\ &= s\left[d(u,z_n) + d(z_n,z_{n+1}) + d(Tz_n,Tu)\right] \\ &\le s\left[d(u,z_n) + d(z_n,z_{n+1}) + \lambda\left[d(z_n,Tz_n)+d(u,Tu)\right]\right] \\ &\le sd(u,z_n) + sd(z_n,z_{n+1}) + s\lambda d(z_n,z_{n+1}) + s\lambda d(u,Tu) \\ &\le s\lim_{n\to\infty} d(u,z_n) + s\lim_{n\to\infty} d(z_n,z_{n+1}) + s\lambda\lim_{n\to\infty} d(z_n,z_{n+1}) + s\lambda d(u,Tu) \\ &\le s\lambda d(u,Tu), \text{ by } (3.11) \text{ and } (3.12) \end{aligned}$$

Thus $\quad 0 \le d(u,Tu) \le s\lambda d(u,Tu); \;\; s\lambda < 1.$

It follows from above inequality that $d(u,Tu) = 0$, i.e. , $Tu = u$. Thus u is a fixed point of T. For uniqueness, let v be another fixed point of T. Then it follows that

$$0 \le d(u,v) = d(Tu,Tv) \le \lambda\left[d(u,Tu)+d(v,Tv)\right] = \lambda\left[d(u,u)+d(v,v)\right] = 0$$

a contradiction. Therefore, we must have $d(u,v) = 0$, i.e., $u = v.$ Thus fixed point is unique.

Theorem 3. Let (X,d) be a complete complex valued rectangular $b-$metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:

$$d(Tx,Ty) \le k\max\{d(x,y), d(x,Tx), d(y,Ty)\} \qquad (3.13)$$

for all $x, y \in X$, where $k \in \left[0, \dfrac{2}{s+1}\right]$. Then T has a unique fixed point.

Proof Let $z_0 \in X$ and each $n \in \mathbb{N}$, we define $z_n = Tz_{n-1}$; *for all* $n \in \mathbb{N}$.

We consider,

$$\begin{aligned} d(z_1,z_2) &= d(Tz_0,Tz_1) \\ &\le k\max\{d(z_0,z_1), d(z_0,Tz_0), d(z_1,Tz_1)\} \\ &\le k\max\{d(z_0,z_1), d(z_0,z_1), d(z_1,z_2)\} \\ &= k\max\{d(z_0,z_1), d(z_1,z_2)\} \\ &\le k\left[d(z_0,z_1) + d(z_1,z_2)\right], \end{aligned}$$

$$d(z_1,z_2) \le kd(z_0,z_1) + kd(z_1,z_2). \qquad (3.14)$$

$$\Rightarrow \; d(z_1,z_2) \le \left(\frac{k}{1-k}\right) d(z_0,z_1). \qquad (3.15)$$

Thus, $d(z_1,z_2) \le \left(\frac{k}{1-k}\right) e_0$, *for some* $k \in \left[0,\ \frac{2}{s+1}\right]$, *where* $e_0 = d(z_0,z_1)$.

$$
\begin{aligned}
d(z_2,z_3) &= d(Tz_1,Tz_2) \\
&\le k \max\{d(z_1,z_2), d(z_1,Tz_1), d(z_2,Tz_2)\} \\
&\le k \max\{d(z_1,z_2), d(z_1,z_2)\ ,\ d(z_2,z_3)\} \\
&= k \max\{d(z_1,z_2), d(z_2,z_3)\} \\
&\le k\left[d(z_1,z_2) + d(z_2,z_3)\right],
\end{aligned}
$$

$$d(z_2,z_3) \le kd(z_1,z_2) + kd(z_2,z_3). \qquad (3.16)$$

$$\Rightarrow d(z_2,z_3) \le \left(\frac{k}{1-k}\right) d(z_1,z_2). \qquad (3.17)$$

Thus, $d(z_2,z_3) \le \left(\frac{k}{1-k}\right) e_1$, *for some* $k \in \left[0,\ \frac{2}{s+1}\right]$, *where* $e_1 = d(z_1,z_2)$.

Using (3.15) and (3.17) we have

$$
\begin{aligned}
d(z_2,z_3) &\le \left(\frac{k}{1-k}\right) d(z_1,z_2), \\
&\le \left(\frac{k}{1-k}\right)\left(\frac{k}{1-k}\right) d(z_0,z_1)
\end{aligned}
$$

$$d(z_2,z_3) = \left(\frac{k}{1-k}\right)^2 e_0,\ \textit{for some}\ k \in \left[0,\ \frac{2}{s+1}\right] \qquad (3.18)$$

In general, we obtain that

$$d(z_n,z_{n+1}) \le \left(\frac{k}{1-k}\right)^n d(z_0,z_1)\ \textit{for some}\ \ k \in \left[0,\ \frac{2}{s+1}\right] \textit{where}\ e_0 = d(z_0,z_1). \qquad (3.19)$$

Let $e_n = d(z_n,z_{n+1})$, $\forall n \in \mathbb{N}$

$$
\begin{aligned}
d(z_n,z_{n+2}) \;&=\; d(Tz_{n-1},Tz_{n+1}) \\
&\le k \max\{d(z_{n-1},z_{n+1}), d(z_{n-1},Tz_{n-1}), d(z_{n+1},Tz_{n+1})\} \\
&\le k \max\{d(z_{n-1},z_{n+1}), d(z_{n-1},z_n)\ ,\ d(z_{n+1},z_{n+2})\} \\
&\le k \max\{s[d(z_{n-1},z_n) + d(z_n,z_{n+2}) + d(z_{n+2},z_{n+1})], d(z_{n-1},z_n)\ ,\ d(z_{n+1},z_{n+2})\} \\
&\le k \max\{sd(z_{n-1},z_n) + sd(z_n,z_{n+2}) + sd(z_{n+2},z_{n+1}), d(z_{n-1},z_n)\ ,\ d(z_{n+1},z_{n+2})\} \\
&= k\left[sd(z_{n-1},z_n) + sd(z_n,z_{n+2}) + sd(z_{n+2},z_{n+1})\right],
\end{aligned}
$$

$$d(z_n,z_{n+2}) \le ksd(z_{n-1},z_n) + ksd(z_n,z_{n+2}) + ksd(z_{n+2},z_{n+1}). \qquad (3.20)$$

$$\Rightarrow \qquad d(z_n,z_{n+2}) \le ksd(z_{n-1},z_n) + ksd(z_n,z_{n+2}) + ksd(z_{n+2},z_{n+1})$$

$$\Rightarrow\ d(z_n,z_{n+2}) - ksd(z_n,z_{n+2}) \le ksd(z_{n-1},z_n) + ksd(z_n,z_{n+2})$$

$$\Rightarrow \qquad [1-ks]\,d(z_n,z_{n+2}) \le ks\left[d(z_{n-1},z_n) + d(z_n,z_{n+2})\right]$$

$$\Rightarrow \quad d(z_n, z_{n+2}) \le \left(\frac{ks}{1-ks}\right)\left[d(z_{n-1}, z_n) + d(z_n, z_{n+2})\right]$$

$$\Rightarrow \quad d(z_n, z_{n+2}) \le \left(\frac{ks}{1-ks}\right)\left[\left(\frac{k}{1-k}\right)^{n-1} e_0 + \left(\frac{k}{1-k}\right)^{n} e_0\right]$$

$$\Rightarrow \quad d(z_n, z_{n+2}) \le \left(\frac{k}{1-k}\right)^{n-1}\left[\left(\frac{ks}{1-ks}\right)e_0 + \left(\frac{ks}{1-ks}\right)\left(\frac{k}{1-k}\right)e_0\right]$$

$$\Rightarrow \quad d(z_n, z_{n+2}) \le \left(\frac{k}{1-k}\right)^{n-1} e^*_0, \text{ where } e^*_0 = \left[\left(\frac{ks}{1-ks}\right)e_0 + \left(\frac{ks}{1-ks}\right)\left(\frac{k}{1-k}\right)^{2} e_0\right]$$

$$\text{for some } k \in \left[0, \frac{2}{s+1}\right] \qquad (3.21)$$

$$\begin{aligned}
d(z_n, z_{n+2m+1}) &\le s\left[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + d(z_{n+2}, z_{n+2m+1})\right] \\
&\le s\Big[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) \\
&\quad + s\left[d(z_{n+2}, z_{n+3}) + d(z_{n+3}, z_{n+4}) + d(z_{n+4}, z_{n+2m+1})\right]\Big] \\
&\le s\left[(e_n + e_{n+1}) + s\left[(e_{n+2} + e_{n+3}) + s\left[(e_{n+4} + e_{n+5}) + \dots\right.\right.\right. \\
&\quad + s\left[(e_{n+2m-2} + e_{n+2m-1} + e_{n+2m})\right. \\
&\le se_n + se_{n+1} + s^2 e_{n+2} + s^2 e_{n+3} + s^3 e_{n+4} + s^3 e_{n+5} + \dots \\
&\quad + s^m e_{n+2m-2} + s^m e_{n+2m-1} + s^m e_{n+2m}
\end{aligned}$$

$$\begin{aligned}
&\le s\left[\left(\frac{k}{1-k}\right)^{n} e_0 + \left(\frac{k}{1-k}\right)^{n+1} e_0\right] + s^2\left[\left(\frac{k}{1-k}\right)^{n+2} e_0 + \left(\frac{k}{1-k}\right)^{n+3} e_0\right] \\
&\quad + s^3\left[\left(\frac{k}{1-k}\right)^{n+4} e_0 + \left(\frac{k}{1-k}\right)^{n+5} e_0\right] + \dots \\
&\quad + s^m\left[\left(\frac{k}{1-k}\right)^{n+2m-2} e_0 + \left(\frac{k}{1-k}\right)^{n+2m-1} e_0 + \left(\frac{k}{1-k}\right)^{n+2m} e_0\right] \\
&\le \Bigg[s\left(\frac{k}{1-k}\right)^{n} e_0 + s^2\left(\frac{k}{1-k}\right)^{n+2} e_0 + s^3\left(\frac{k}{1-k}\right)^{n+4} e_0 + \dots \\
&\quad + s^m\left(\frac{k}{1-k}\right)^{n+2m-2} e_0 + s^m\left(\frac{k}{1-k}\right)^{n+2m} e_0\Bigg] \\
&\quad + \left[s\left(\frac{k}{1-k}\right)^{n+1} e_0 + s^2\left(\frac{k}{1-k}\right)^{n+3} e_0 + s^3\left(\frac{k}{1-k}\right)^{n+5} e_0 + \dots + s^m\left(\frac{k}{1-k}\right)^{n+2m-1} e_0\right] \\
&\le s\left(\frac{k}{1-k}\right)^{n} e_0\left[1 + s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + \dots + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-2} + s^{m-1}\left(\frac{k}{1-k}\right)^{2m}\right] \\
&\quad + s\left(\frac{k}{1-k}\right)^{n+1} e_0\left[1 + s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + \dots + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-2}\right]
\end{aligned}$$

$$\leq s\left(\frac{k}{1-k}\right)^{n} e_0\left[1+s\left(\frac{k}{1-k}\right)^{2}+s^{2}\left(\frac{k}{1-k}\right)^{4}+\ldots\right]+s\left(\frac{k}{1-k}\right)^{n+1} e_0\left[1+s\left(\frac{k}{1-k}\right)^{2}+s^{2}\left(\frac{k}{1-k}\right)^{4}+\ldots\right]$$

$$\leq\left[1+s\left(\frac{k}{1-k}\right)^{2}+s^{2}\left(\frac{k}{1-k}\right)^{4}+\ldots\right]\left[s\left(\frac{k}{1-k}\right)^{n} e_0+s\left(\frac{k}{1-k}\right)^{n+1} e_0\right]$$

$$=\frac{1}{1-s\left(\frac{k}{1-k}\right)^{2}}\left[s\left(\frac{k}{1-k}\right)^{n} e_0+s\left(\frac{k}{1-k}\right)^{n+1} e_0\right]$$

$$=\frac{s\left(\frac{k}{1-k}\right)^{n} e_0\left(1+\left(\frac{k}{1-k}\right)\right)}{1-s\left(\frac{k}{1-k}\right)^{2}},$$

$$d\left(z_n,z_{n+2m+1}\right)\leq\frac{s\left(\frac{k}{1-k}\right)^{n} e_0\left(1+\left(\frac{k}{1-k}\right)\right)}{1-s\left(\frac{k}{1-k}\right)^{2}}. \qquad (3.22)$$

$$\begin{aligned}
d\left(z_n,z_{n+2m}\right) &\leq s\left[d\left(z_n,z_{n+1}\right)+d\left(z_{n+1},z_{n+2}\right)+d\left(z_{n+2},z_{n+2m}\right)\right]\\
&\leq s\left[d\left(z_n,z_{n+1}\right)+d\left(z_{n+1},z_{n+2}\right)\right)\\
&\quad+s\left[d\left(z_{n+2},z_{n+3}\right)+d\left(z_{n+3},z_{n+4}\right)+d\left(z_{n+4},z_{n+2m}\right)\right]\\
&\leq s\left[\left(e_n+e_{n+1}\right)+s\left[\left(e_{n+2}+e_{n+3}\right)+s\left[\left(e_{n+4}+e_{n+5}\right)+\ldots\right.\right.\right.\\
&\quad+s^{m-1}\left[\left(e_{2m-4}+e_{2m-3}\right)+s^{m-1}d\left(z_{n+2m-2},z_{n+2m}\right)\right.\\
&\leq se_n+se_{n+1}+s^{2}e_{n+2}+s^{2}e_{n+3}+s^{3}e_{n+4}+s^{3}e_{n+5}+\ldots\\
&\quad+s^{m-1}e_{2m-4}+s^{m-1}e_{2m-3}+s^{m-1}d\left(z_{n+2m-2},z_{n+2m}\right)\\
&\leq s\left[\left(\frac{k}{1-k}\right)^{n} e_0+\left(\frac{k}{1-k}\right)^{n+1} e_0\right]+s^{2}\left[\left(\frac{k}{1-k}\right)^{n+2} e_0+\left(\frac{k}{1-k}\right)^{n+3} e_0\right]\\
&\quad+s^{3}\left[\left(\frac{k}{1-k}\right)^{n+4} e_0+\left(\frac{k}{1-k}\right)^{n+5} e_0\right]+\ldots+s^{m-1}\left[\left(\frac{k}{1-k}\right)^{2m-4} e_0+\left(\frac{k}{1-k}\right)^{2m-3}\right]\\
&\quad+s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*}
\end{aligned}$$

$$\leq \left[s\left(\frac{k}{1-k}\right)^{n} e_0 + s^2\left(\frac{k}{1-k}\right)^{n+2} e_0 + s^3\left(\frac{k}{1-k}\right)^{n+4} e_0 + \ldots + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-4} e_0 \right] +$$

$$+ \left[s\left(\frac{k}{1-k}\right)^{n+1} e_0 + s^2\left(\frac{k}{1-k}\right)^{n+3} e_0 + s^3\left(\frac{k}{1-k}\right)^{n+5} e_0 + \ldots + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-3} e_0 \right]$$

$$+ s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*}$$

$$\leq s\left(\frac{k}{1-k}\right)^{n} e_0 \left[1 + s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + \ldots + s^{m-2}\left(\frac{k}{1-k}\right)^{2m-n-4} \right]$$

$$+ s\left(\frac{k}{1-k}\right)^{n+1} e_0 \left[1 + s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + \ldots + s^{m-2}\left(\frac{k}{1-k}\right)^{2m-n-4} \right]$$

$$+ s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*}$$

$$\leq s\left(\frac{k}{1-k}\right)^{n} e_0 \left[1 + s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + \ldots \right] + s\left(\frac{k}{1-k}\right)^{n+1} e_0 \left[1 + s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + \ldots \right]$$

$$+ s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*}$$

$$\leq \left[1 + s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + \ldots \right] \left[s\left(\frac{k}{1-k}\right)^{n} e_0 + s\left(\frac{k}{1-k}\right)^{n+1} e_0 \right] + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*}$$

$$= \frac{1}{1 - s\left(\frac{k}{1-k}\right)^{2}} \left[s\left(\frac{k}{1-k}\right)^{n} e_0 + s\left(\frac{k}{1-k}\right)^{n+1} e_0 \right] + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*}$$

$$= \frac{s\left(\frac{k}{1-k}\right)^{n} e_0 \left(1 + \left(\frac{k}{1-k}\right)\right)}{1 - s\left(\frac{k}{1-k}\right)^{2}} + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*},$$

$$d\left(z_n, z_{n+2m}\right) \leq \frac{s\left(\frac{k}{1-k}\right)^{n} e_0 \left(1 + \left(\frac{k}{1-k}\right)\right)}{1 - s\left(\frac{k}{1-k}\right)^{2}} + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^{*}. \qquad (3.23)$$

It follows from (3.22) and (3.23) that

$$\lim_{n\to\infty} d\left(z_n, z_{n+p}\right) = 0 \text{ for all } p > 0. \qquad (3.24)$$

Thus $\{z_n\}$ is a Cauchy sequence in X. By completeness of (X,d) there exists $u \in X$ such that

$$\lim_{n\to\infty} z_n = u. \qquad (3.25)$$

We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,Tu) &\le s\left[d(u,z_n) + d(z_n,z_{n+1}) + d(z_{n+1},Tu)\right] \\ &= s\left[d(u,z_n) + d(z_n,z_{n+1}) + d(Tz_n,Tu)\right] \\ &\le s\left[d(u,z_n) + d(z_n,z_{n+1}) + k\max\{d(z_n,u),d(z_n,Tz_n),d(u,Tu)\}\right] \\ &\le s\left[d(u,z_n) + d(z_n,z_{n+1}) + k\max\{d(z_n,u),d(z_n,z_{n+1}),d(u,Tu)\}\right] \\ &\le s\left[d(u,z_n) + d(z_n,z_{n+1}) + k\{d(z_n,u)+d(z_n,z_{n+1})+d(u,Tu)\}\right] \\ &\le sd(u,z_n) + sd(z_n,z_{n+1}) + skd(z_n,u)+skd(z_n,z_{n+1})+skd(u,Tu) \\ &\le \left[sd(u,z_n)+skd(z_n,u)\right] + \left[sd(z_n,z_{n+1})+skd(z_n,z_{n+1})\right]+skd(u,Tu) \\ &\le [1+k]sd(z_n,u)+[1+k]sd(z_n,z_{n+1}) + skd(u,Tu) \\ &\le [1+k]s\lim_{n\to\infty} d(z_n,u)+[1+k]s\lim_{n\to\infty} d(z_n,z_{n+1})+skd(u,Tu) \end{aligned}$$

$$d(u,Tu) \le skd(u,Tu). \text{by} (3.24) \text{ and } (3.25)$$

Thus $0 \le d(u,Tu) \le skd(u,Tu);\ \ sk<1.$

It follows from above inequality that $d(u,Tu) = 0$, i.e. , $Tu = u$. Thus u is a fixed point of T. For uniqueness, let v be another fixed point of T. Then it follows that

$d(u,v) = d(Tu,Tv) \le k\max\{d(u,v),d(u,Tu), d(v,Tv)\} \le kd(u,v)$. Thus a contradiction. Therefore, we must have $d(u,v) = 0$, i.e., $u = v$. Thus fixed point is unique.

4. Conclusions

The purpose of this paper is to study the existence of fixed point for self mappings under generalized kannan mappings type concept in b-metric spaces. Our result extend and generalize the result derived by Ege [21] and many others.Moreover, we give examples as a satisfying the theorems in complex valued rectangular b-metric spaces as follows:

Theorem 1. Let (X,d) be a complete complex valued rectangular $b-$metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:

$$d(Tx,Ty) \le \lambda\left[d(x,Tx) + d(y,Ty)\right]$$

for all $x,y \in X$, where $\lambda \in \left[0, \dfrac{2}{s+1}\right]$. Then T has a unique fixed point.

Theorem 2. Let (X,d) be a complete complex valued rectangular $b-$metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:

$$d(Tx,Ty) \le k\max\{d(x,y),d(x,Tx), d(y,Ty)\}$$

for all $x,y \in X$, where $k \in \left[0, \dfrac{2}{s+1}\right]$. Then T has a unique fixed point.

5. Discussion

Future research directions may also be possible.

Open problems 1:

If T satisfies $(d(Tx,Ty) \le \lambda[d(x,y)+d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx)])$ then T has a unique fixed point.

Open problems 2:

If T satisfies $(d(Tx,Ty) \leq k \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\})$ then T has a unique fixed point.

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