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Base-β Representations and Irreducibility of Polynomials over Any Imaginary Quadratic Field

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ABSTRACT

Let K be an imaginary quadratic field whose ring of integers O_K is a Euclidean domain. In the earlier work, the so-called base-β representation for nonzero elements of O_K was constructed and the irreducibility criterion for polynomials in $O_K[x]$ was established, namely if $\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$ is a base-β representation of a prime element $\pi \in O_K$ and the digits α_{n-1} and α_n satisfy some natural restrictions, then the polynomial $f(x)$ is irreducible in $O_K[x]$. A generalization of this criterion was also verified by considering $\omega\pi$ ($\omega \in O_K \setminus \{0\}$) instead of π. In this paper, we extend these results to any imaginary quadratic field K.

Keywords: Gauss's lemma; Imaginary quadratic field; Irreducible element; Irreducible polynomial; Ring of integers

1. Introduction

One of the popular topics in the development of number theory is the relationship between prime numbers and irreducible polynomials. The problem of determining the irreducibility of a polynomial in $\mathbb{Z}[x]$ has been extensively studied and has become an interesting topic in mathematics. There are many irreducibility criteria for polynomials in $\mathbb{Z}[x]$. Here, we are interested in an elegant result of A. Cohn, given by Pólya and Szegö [1]: if

$$p = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$$

is the base-10 representation of a prime number p, then the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is irreducible in $\mathbb{Z}[x]$. For example, one can immediately

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infer that $f(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 3$ is irreducible in $\mathbb{Z}[x]$ due to the decimal representation of the prime number 54323. In 1981, Brillhart et al. [2] generalized this result to any arbitrary base b. On the same matter, Murty [3] provided another proof of this result using a conceptually simpler argument than the one in [2]. In 1982, Filaseta [4] generalized this result in another way by considering wp instead of p: if

$$wp = b_m b^m + b_{m-1} b^{m-1} + \cdots + b_1 b + b_0$$

is the base-b representation of wp, where w and b are positive integers, $w < b$, and p is a prime number, then the polynomial $f(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ is irreducible over $\mathbb{Q}$.

The irreducibility criteria for polynomials in $\mathbb{Z}[x]$ mentioned earlier motivate us to study and establish irreducibility criteria for polynomials in $O_K[x]$, where K is any imaginary quadratic field. First of all, let us summarize some information on a quadratic field taken from [5] as follows: for any quadratic field K, there exists a unique square-free integer $m \neq 1$ such that

$$K = \mathbb{Q}\left(\sqrt{m}\right) = \left\{ r + s\sqrt{m} \mid r, s \in \mathbb{Q} \right\}.$$

The field K is said to be *real* if $m > 0$ and *imaginary* if $m < 0$. The set of algebraic integers that lie in K is denoted by O_K, called the *ring of integers* of K. In fact,

$$O_K = \{ a + b\sigma_m \mid a, b \in \mathbb{Z} \},$$

where

$$\sigma_m := \begin{cases} \sqrt{m} & \text{if } m \not\equiv 1 \pmod 4, \\ \dfrac{1 + \sqrt{m}}{2} & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

In particular, when $m = -1$, one can see that $O_K = \mathbb{Z}[i]$, the ring of *Gaussian integers*. Note that K is the quotient field of O_K, which is an integral domain and the set of units in the polynomial ring $O_K[x]$ is $U(O_K)$, the group of units in O_K.

For $\alpha, \beta \in O_K$ with $\alpha \neq 0$, we say that α *divides* β, denoted by $\alpha \mid \beta$, if there exists $\delta \in O_K$ such that $\beta = \alpha\delta$. We write $\alpha \nmid \beta$ to indicate that β is not divisible by α. A polynomial $p(x) \in O_K[x]$ is *irreducible* in $O_K[x]$ if it is an irreducible element of the ring $O_K[x]$; in other words, $p(x)$ is neither a zero polynomial nor a unit in O_K and if, whenever $p(x) = f(x)g(x)$ with $f(x)$ and $g(x)$ in $O_K[x]$, then either $f(x)$ or $g(x)$ is a unit in O_K. Polynomials that are not irreducible are said to be *reducible*. In addition, if O_K is a unique factorization domain and $f(x)$ is irreducible in $O_K[x]$, then $f(x)$ is irreducible over K [6].

For $\beta = a + b\sigma_m \in O_K$, we denote the *norm* of β by $N(\beta) =$

$$\begin{cases} a^2 - mb^2 & \text{if } m \not\equiv 1 \pmod 4, \\ a^2 + ab + b^2\left(\dfrac{1-m}{4}\right) & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

We note from [5] that if $N(\beta) = \pm p$, where p is a rational prime, then β is an irreducible element of O_K. One can see that a prime element of O_K is always irreducible, while the converse holds if O_K is a unique factorization domain. Moreover, if K is an imaginary quadratic field, then $|\beta|^2 = N(\beta) \in \mathbb{N}$ for all $\beta \in O_K \backslash \{0\}$ and $|\beta| = 1$ for all $\beta \in U(O_K)$ [5].

In 2017, Singthongla et al. [8] constructed the so-called *base-β representation* for nonzero elements of O_K and applied it to establish the result of A. Cohn in $O_K[x]$, where $K = \mathbb{Q}\left(\sqrt{m}\right)$ is an imaginary quadratic field such that O_K is a Euclidean domain, namely when $m = -1, -2, -3, -7, -11$ [5]. They also determined the *base-$\beta(C)$ representation* for nonzero Gaussian integers and established irreducibility criteria for polynomials in $\mathbb{Z}[i][x]$, where $C = \{x + yi \mid x = 0, 1, \ldots, ((a^2+b^2)/d) - 1$ and $y = 0, 1, \ldots, d-1\}$ is a complete residue system modulo $\beta \in \mathbb{Z}[i]$ [7]. In their work, the following results were proved:

Theorem 1.1. *Let $K = \mathbb{Q}(\sqrt{m})$ where $m = -1, -2, -3, -7, -11$, and let*

$$A_m = \begin{cases} \dfrac{\sqrt{1-m}}{2} & \text{if } m = -1, -2, \\ \dfrac{\sqrt{4-m}}{4} & \text{if } m = -3, -7, -11. \end{cases}$$

Let $\beta \in O_K$ be such that $|\beta| > 1 + A_m$. Then any $\eta \in O_K \backslash \{0\}$ has a base-β representation in the form

$$\eta = \alpha_n \beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1 \beta + \alpha_0,$$

where $n \geq 0$, $\alpha_n \in O_K \backslash \{0\}$, $|\alpha_n| < |\beta|$, and $\alpha_i \in O_K$ with $0 \leq |\alpha_i| \leq A_m|\beta|$ for all $i \in \{0, 1, \ldots, n-1\}$.

Theorem 1.2. *Let $K = \mathbb{Q}(\sqrt{m})$ where $m = -1, -2, -3, -7, -11$, and let $B_m =$*

$$\begin{cases} \left(6 + \sqrt{2} + \sqrt{6 + 12\sqrt{2}}\right)/4 \approx 3.05 & \text{if } m = -1, \\ \left(6 + \sqrt{3} + \sqrt{7 + 12\sqrt{3}}\right)/4 \approx 3.2508 & \text{if } m = -2, \\ \left(12 + \sqrt{7} + \sqrt{23 + 24\sqrt{7}}\right)/8 \approx 2.99327 & \text{if } m = -3, \\ \left(12 + \sqrt{11} + \sqrt{27 + 24\sqrt{11}}\right)/8 \approx 3.20516 & \text{if } m = -7, \\ \left(12 + \sqrt{15} + \sqrt{31 + 24\sqrt{15}}\right)/8 \approx 3.37579 & \text{if } m = -11. \end{cases}$$

Let $\beta \in O_K$ be such that $|\beta| \geq B_m$ and $\mathrm{Re}(\beta) \geq 1$. For a prime element π of O_K, if

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

is its base-β representation with $n \geq 1$ satisfying the conditions $\mathrm{Re}(\alpha_n) \geq 1$, $\mathrm{Re}(\alpha_{n-1}) \geq 0$, $\mathrm{Im}(\alpha_{n-1}) \geq 0$ and $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible in $O_K[x]$.

Denote $C' := \{x + yi \mid x = 0, 1, \ldots, \max\{|a|, |b|\} - 1$ and $y = 0, 1, \ldots, d-1\} \subseteq C$.

Theorem 1.3. *Let $\beta \in \{2 \pm 2i, 1 \pm 3i, 3 \pm i\}$ or $\beta = a + bi \in \mathbb{Z}[i]$ be such that $|\beta| \geq 2 + \sqrt{2}$ and $a \geq 1$. For a Gaussian prime π, if*

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta),$$

with $n \geq 1$, $\mathrm{Re}(\alpha_n) \geq 1$, and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in C'$ satisfying $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible in $\mathbb{Z}[i][x]$.

Theorem 1.4. *If π is a Gaussian prime such that*

$$\pi = \alpha_n 3^n + \alpha_{n-1}3^{n-1} + \cdots + \alpha_1 3 + \alpha_0,$$

where $n \geq 3$, $\mathrm{Re}(\alpha_n) \geq 1$, and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in C'$ satisfying the conditions

$$\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1}),$$

$$\mathrm{Re}(\alpha_{n-2})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-2}),$$

$$\mathrm{Re}(\alpha_{n-2})\,\mathrm{Im}(\alpha_{n-1}) \geq \mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_{n-2}),$$

then the polynomial $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0$ is irreducible in $\mathbb{Z}[i][x]$.

Afterward, Kanasri et al. [9] proved the following theorem, which is a generalization of Theorem 1.2 when considering $\omega\pi$ ($\omega \in O_K \backslash \{0\}$) instead of π.

Theorem 1.5. *Let $K = \mathbb{Q}\left(\sqrt{m}\right)$ where $m = -1, -2, -3, -7, -11$, $\omega \in O_K\backslash\{0\}$, and let* $B_m(\omega) =$

$$\begin{cases} \left(2(2|\omega|+1)+\sqrt{2}+\sqrt{6+4(2|\omega|+1)\sqrt{2}}\right)/4 & \text{if } m=-1, \\ \left(2(2|\omega|+1)+\sqrt{3}+\sqrt{7+4(2|\omega|+1)\sqrt{3}}\right)/4 & \text{if } m=-2, \\ \left(4(2|\omega|+1)+\sqrt{7}+\sqrt{23+8(2|\omega|+1)\sqrt{7}}\right)/8 & \text{if } m=-3, \\ \left(4(2|\omega|+1)+\sqrt{11}+\sqrt{27+8(2|\omega|+1)\sqrt{11}}\right)/8 & \text{if } m=-7, \\ \left(4(2|\omega|+1)+\sqrt{15}+\sqrt{31+8(2|\omega|+1)\sqrt{15}}\right)/8 & \text{if } m=-11. \end{cases}$$

Let $\beta \in O_K$ be such that $|\beta| \geq B_m(\omega)$ and $\mathrm{Re}(\beta) \geq |\omega|$. For a prime element π of O_K, if

$$\omega\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

is its base-β representation with $n \geq 1$ satisfying the conditions $\mathrm{Re}(\alpha_n) \geq 1$, $\mathrm{Re}(\alpha_{n-1}) \geq 0$, $\mathrm{Im}(\alpha_{n-1}) \geq 0$ and $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$, then $f(x)$ is irreducible over K.

For any quadratic field $K = \mathbb{Q}\left(\sqrt{m}\right)$, Tadee et al. [10] proved for the case $m \not\equiv 1 \pmod 4$ that the set

$$C = \{x + y\sigma_m \mid x = 0, 1, \ldots, (|N(\beta)|/d) - 1 \text{ and } y = 0, 1, \ldots, d-1\} \quad (1.1)$$

is a complete residue system modulo β, abbreviated by $CRS(\beta)$, where $\beta = a + b\sigma_m \in O_K$ with $d = \gcd(a, b)$. In 2021, Phetnun et al. [11] verified that Eq. (1.1) is also a $CRS(\beta)$ for the case $m \equiv 1 \pmod 4$ and the set $C' := \{x + y\sigma_m \mid x = 0, 1, \ldots, \max\{|a|, |b|\} - 1$ and $y = 0, 1, \ldots, d-1\} \subseteq C$ for any $m < 0$. Moreover, they determined the so-called *base-$\beta(C)$ representation* for nonzero elements of O_K and extended Theorem 1.3 to any imaginary quadratic field using such a representation (Theorems 3-4 in [11]). We note that $\eta \in O_K\backslash\{0\}$ has a *base-$\beta(C)$ representation* if

$$\eta = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0, \quad (1.2)$$

for some $n \geq 1$, $\alpha_n \in O_K\backslash\{0\}$ and $\alpha_i \in C$ $(i = 0, 1, \ldots, n-1)$. If $\alpha_i \in C'$ $(i = 0, 1, \ldots, n-1)$, then Eq. (1.2) is called a *base-$\beta(C')$ representation* of η.

Recently, Phetnun and Kanasri [12] established further irreducibility criteria for polynomials in $O_K[x]$, where K is an imaginary quadratic field, which extended Theorem 1.4 to any imaginary quadratic field.

In similar fashion as the results in [11, 12], we hence aim to extend Theorems 1.1-1.2, and 1.5 to any imaginary quadratic field.

2. Base-β Representations

In this section, we construct a base-β representation for nonzero elements of O_K, where K is an imaginary quadratic field. We start with the division algorithm for elements of O_K.

Proposition 2.1. *Let $K = \mathbb{Q}\left(\sqrt{m}\right)$ be an imaginary quadratic field and let $\beta \in O_K\backslash\{0\}$ be fixed. For $\alpha \in O_K$, there exist $\lambda, \rho \in O_K$ such that $\alpha = \lambda\beta + \rho$ with $0 \leq |\rho| \leq A_m|\beta|$, where*

$$A_m := \begin{cases} \dfrac{\sqrt{1-m}}{2} & \text{if } m \not\equiv 1 \pmod 4, \\ \dfrac{\sqrt{4-m}}{4} & \text{if } m \equiv 1 \pmod 4. \end{cases} \quad (2.1)$$

Proof. Assume that $\alpha/\beta = r + s\sqrt{m}$, where $r, s \in \mathbb{Q}$. We consider two possible cases.
Case 1: $m \not\equiv 1 \pmod 4$. Let $a = \lfloor r + (1/2) \rfloor$ and $b = \lfloor s + (1/2) \rfloor$, where $\lfloor\cdot\rfloor$ is the floor function. Then $|r - a| \leq 1/2$

and $|s-b| \leq 1/2$. Now, we let $\lambda = a + b\sigma_m$ and $\rho = \alpha - \lambda\beta$. Then $\lambda, \rho \in O_K$, $\alpha = \lambda\beta + \rho$, and

$$\begin{aligned} 0 \leq |\rho| &= |\beta| \left| \frac{\alpha}{\beta} - \lambda \right| \\ &= |\beta| \left| (r-a) + (s-b)\sqrt{m} \right| \\ &= |\beta| \sqrt{(r-a)^2 - m\,(s-b)^2} \\ &\leq |\beta| \sqrt{\frac{1-m}{4}} \\ &= A_m |\beta|. \end{aligned}$$

Case 2: $m \equiv 1 \pmod 4$. Let $a = \lfloor 2s + (1/2) \rfloor$ and $b = \lfloor r - (a/2) + (1/2) \rfloor$. It follows that $|2s - a| \leq 1/2$ and $|r - (a/2) - b| \leq 1/2$. Now, we let $\lambda = b + a\sigma_m$ and $\rho = \alpha - \lambda\beta$. Then $\lambda, \rho \in O_K$, $\alpha = \lambda\beta + \rho$, and

$$\begin{aligned} 0 \leq |\rho| &= |\beta| \left| \frac{\alpha}{\beta} - \lambda \right| \\ &= |\beta| \left| \left(r - \frac{a}{2} - b\right) + \left(s - \frac{a}{2}\right)\sqrt{m} \right| \\ &= |\beta| \sqrt{\left(r - \frac{a}{2} - b\right)^2 - m\left(s - \frac{a}{2}\right)^2} \\ &\leq |\beta| \sqrt{\frac{4-m}{16}} \\ &= A_m |\beta|. \end{aligned}$$

This completes the proof. □

Our first main result reads:

Theorem 2.2. *Let $K = \mathbb{Q}\left(\sqrt{m}\right)$ be an imaginary quadratic field and let $\beta \in O_K$ be such that $|\beta| \geq 2$. Then any $\eta \in O_K \backslash \{0\}$ has a base-β representation in the form*

$$\eta = \alpha_n \beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1 \beta + \alpha_0,$$

where $n \geq 0$, $\alpha_i \in O_K$ with $\alpha_n \neq 0$ and $0 \leq |\alpha_i| \leq A_m|\beta|$ for all $i \in \{0, 1, \ldots, n\}$, where A_m is defined as in Eq. (2.1).

Proof. If $|\eta| \leq A_m|\beta|$, then we are done. Now, assume that $|\eta| > A_m|\beta|$. It follows from Proposition 2.1 that

$$\eta = \delta_0 \beta + \alpha_0, \; 0 \leq |\alpha_0| \leq A_m|\beta|. \quad (2.2)$$

Clearly, $\delta_0 \neq 0$. We next claim that $|\eta| > |\delta_0|$. Suppose not, we have $|\delta_0| \geq |\eta| = |\delta_0\beta + \alpha_0| \geq |\delta_0||\beta| - |\alpha_0|$ and so $|\alpha_0| \geq |\delta_0|\left(|\beta| - 1\right)$. Since $|\beta| \geq 2$, we consequently have $|\delta_0| \geq |\eta| > A_m|\beta| \geq |\alpha_0| \geq |\delta_0|\left(|\beta| - 1\right) \geq |\delta_0|$, which is a contradiction.

Returning to Eq. (2.2), if $|\delta_0| \leq A_m|\beta|$, then Eq. (2.2) is a base-β representation of η and so we are done. On the other hand, if $|\delta_0| > A_m|\beta|$, then we continue by dividing δ_0 by β and using the claim to obtain

$$\delta_0 = \delta_1\beta + \alpha_1, \; 0 \leq |\alpha_1| \leq A_m|\beta| \text{ and } |\delta_0| > |\delta_1|. \quad (2.3)$$

Again, it is clear that $\delta_1 \neq 0$. It follows from Eqs. (2.2)- (2.3) that

$$\eta = \delta_1\beta^2 + \alpha_1\beta + \alpha_0, \; 0 \leq |\alpha_i| \leq A_m|\beta| \;\; (i = 0, 1). \quad (2.4)$$

If $|\delta_1| \leq A_m|\beta|$, then the process stops again and Eq. (2.4) is a base-β representation of η. While if $|\delta_1| > A_m|\beta|$, then we continue by dividing δ_1 by β. If this process does not stop, then we obtain an infinite sequence $(\delta_i)_{i \geq 0}$ of elements of O_K such that $|\delta_i| > A_m|\beta| > 0$ for all $i \geq 0$ satisfying

$$\delta_i = \delta_{i+1}\beta + \alpha_{i+1}, \; 0 \leq |\alpha_{i+1}| \leq A_m|\beta| \text{ and }$$
$$|\delta_i| > |\delta_{i+1}| \qquad (i \geq 0).$$

It follows that $(|\delta_i|^2)_{i \geq 0}$ is a strictly decreasing sequence of positive integers, which is a contradiction because there are finitely many positive integers between $|\delta_0|^2$ and 0. Thus there exists the smallest nonnegative integer k such that $|\delta_k| \leq A_m|\beta|$ and

$$\eta = \delta_k\beta^{k+1} + \alpha_k\beta^k + \cdots + \alpha_1\beta + \alpha_0.$$

If $\delta_k = 0$, then $|\delta_{k-1}| = |\delta_k\beta + \alpha_k| = |\alpha_k| \leq A_m|\beta|$, contradicting the property of k. Thus $\delta_k \neq 0$ and so

$$\eta = \alpha_{k+1}\beta^{k+1} + \alpha_k\beta^k + \cdots + \alpha_1\beta + \alpha_0,$$

where $\alpha_{k+1} := \delta_k$ and $0 \leq |\alpha_i| \leq A_m|\beta|$ for all $i \in \{0, 1, \ldots, k+1\}$, as desired. □

For $m \in \{-1, -2, -3, -7, -11\}$, we have that $|\alpha_n| \leq A_m|\beta| < |\beta|$, and the condition $|\beta| \geq 2$ in Theorem 2.2 can be reduced to $|\beta| > 1 + A_m$. This implies that Theorem 1.1 can be extended to any imaginary quadratic field by Theorem 2.2.

According to Theorem 2.2, we illustrate the process for constructing a base-β representation in the following examples.

Example 2.3. Let $K = \mathbb{Q}(\sqrt{-2})$, $\beta = 4$, and $\eta = 2615 + 1281\sqrt{-2}$. Then $A_{-2} = \sqrt{3}/2$ and $|\eta| > (\sqrt{3}/2)|\beta|$ (≈ 3.46). By the proof of Theorem 2.2, we have the following

$$\begin{aligned}
&\eta = (654 + 320\sqrt{-2})\beta + (-1 + \sqrt{-2}),\\
&|-1 + \sqrt{-2}| \leq (\sqrt{3}/2)|\beta|;\\
&654 + 320\sqrt{-2} = (164 + 80\sqrt{-2})\beta - 2,\\
&|-2| \leq (\sqrt{3}/2)|\beta|;\\
&164 + 80\sqrt{-2} = (41 + 20\sqrt{-2})\beta + 0,\\
&|0| \leq (\sqrt{3}/2)|\beta|;\\
&41 + 20\sqrt{-2} = (10 + 5\sqrt{-2})\beta + 1,\\
&|1| \leq (\sqrt{3}/2)|\beta|;\\
&10 + 5\sqrt{-2} = (3 + \sqrt{-2})\beta + (-2 + \sqrt{-2}),\\
&|-2 + \sqrt{-2}| \leq (\sqrt{3}/2)|\beta|.
\end{aligned}$$

One can see that $3+\sqrt{-2}$ is the first quotient such that $|3 + \sqrt{-2}| < (\sqrt{3}/2)|\beta|$, showing the process stops and thus

$$\eta = (3 + \sqrt{-2})\beta^5 + (-2 + \sqrt{-2})\beta^4 + \beta^3 - 2\beta + (-1 + \sqrt{-2})$$

is a base-β representation of η.

Example 2.4. Let $K = \mathbb{Q}(\sqrt{-15})$, $\beta = 2 + \sigma_{-15}$, and $\eta = -236 + 59\sigma_{-15}$. Then $A_{-15} = \sqrt{19}/4$ and $|\eta| > (\sqrt{19}/4)|\beta|$ (≈ 3.45). By the proof of Theorem 2.2, we have the following

$$\begin{aligned}
&\eta = (-47 + 35\sigma_{-15})\beta + (-2 + \sigma_{-15}),\\
&|-2 + \sigma_{-15}| \leq (\sqrt{19}/4)|\beta|;\\
&-47 + 35\sigma_{-15} = (12\sigma_{-15})\beta + (1 - \sigma_{-15}),\\
&|1 - \sigma_{-15}| \leq (\sqrt{19}/4)|\beta|;\\
&12\sigma_{-15} = (5 + 2\sigma_{-15})\beta + (-2 + \sigma_{-15}),\\
&|-2 + \sigma_{-15}| \leq (\sqrt{19}/4)|\beta|;\\
&5 + 2\sigma_{-15} = 2\beta + 1,\ |1| \leq (\sqrt{19}/4)|\beta|.
\end{aligned}$$

Since 2 is the first quotient such that $|2| < (\sqrt{19}/4)|\beta|$, the process stops and so

$$\eta = 2\beta^4 + \beta^3 + (-2 + \sigma_{-15})\beta^2 + (1 - \sigma_{-15})\beta + (-2 + \sigma_{-15})$$

is a base-β representation of η.

From Example 2.3, we have that

$$\eta = (3 + \sqrt{-2})\beta^5 + (-2 + \sqrt{-2})\beta^4 + \beta^3 - 2\beta + (-1 + \sqrt{-2})$$

is a base-β representation of η. In another way, we can verify that

$$\eta = (2 + \sqrt{-2})\beta^5 + (3 + \sqrt{-2})\beta^4 - 3\beta^3 - 2\beta + (-1 + \sqrt{-2}) \tag{2.5}$$

is also a base-β representation of η. This shows that a base-β representation of η in O_K is not unique.

3. An Irreducibility Criterion for Polynomials over Any Imaginary Quadratic Field

In this section, we extend Theorem 1.2 to any imaginary quadratic field K. We use irreducible elements instead of prime elements of O_K to make a more general result. We first recall the following useful lemma from [8].

Lemma 3.1. *Let $f(x) = \alpha_n x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{C}[x]$ be such that $n \geq 2$ and $|\alpha_i| \leq M$ $(0 \leq i \leq n-2)$ for some positive real number M. If $f(x)$ satisfies*

(i) $\mathrm{Re}(\alpha_n) \geq 1$, $\mathrm{Re}(\alpha_{n-1}) \geq 0$, *and* $\mathrm{Im}(\alpha_{n-1}) \geq 0$,

(ii) $\mathrm{Re}(\alpha_{n-1})\,\mathrm{Im}(\alpha_n) \geq \mathrm{Re}(\alpha_n)\,\mathrm{Im}(\alpha_{n-1})$,

then any complex zero α of $f(x)$ satisfies either $\mathrm{Re}(\alpha) < 0$ *or* $|\alpha| < \left(1+\sqrt{1+4M}\right)/2$.

We now proceed to our second main result and start with a lemma, which shows that Lemma 3.1 also holds for a linear polynomial in $\mathbb{C}[x]$.

Lemma 3.2. *If $f(x) = \alpha_1 x + \alpha_0$ is a linear polynomial in $\mathbb{C}[x]$ satisfying the conditions (i) and (ii) of Lemma 3.1, then a complex zero α of $f(x)$ satisfies either $\alpha = 0$ or* $\mathrm{Re}(\alpha) < 0$.

Proof. It is clear that $\alpha = -\alpha_0/\alpha_1$ is a complex zero of $f(x)$. If $\alpha_0 = 0$, then $\alpha = 0$. Now, we assume that $\alpha_0 \neq 0$ and let $\alpha_1 = a_1 + b_1 i$, $\alpha_0 = a_0 + b_0 i$. Thus,

$$\begin{aligned}\alpha &= -\frac{a_0 + b_0 i}{a_1 + b_1 i} \\ &= -\frac{(a_0a_1 + b_0b_1) + (a_1b_0 - a_0b_1)i}{a_1^2 + b_1^2},\end{aligned}$$

so $\mathrm{Re}(\alpha) = -(a_0a_1+b_0b_1)/(a_1^2+b_1^2)$. Note that $a_1^2 + b_1^2 > 0$ because $\alpha_1 \neq 0$. By the assumption, we have $a_1 \geq 1$, $a_0 \geq 0$, $b_0 \geq 0$, and $a_0b_1 \geq a_1b_0$. If $a_0 = 0$, then $b_0 = 0$, which is impossible because $\alpha_0 \neq 0$. Thus $a_0 \geq 1$ and also $b_1 \geq 0$. It follows that $a_0a_1 + b_0b_1 > 0$, yielding $\mathrm{Re}(\alpha) < 0$, as desired. □

Theorem 3.3. *Let $K = \mathbb{Q}\left(\sqrt{m}\right)$ be an imaginary quadratic field and $B_m =$*

$$\begin{cases} \left(6 + \sqrt{1-m} + \sqrt{5 - m + 12\sqrt{1-m}}\right)/4 & \text{if } m \not\equiv 1 \pmod 4, \\ \left(12 + \sqrt{4-m} + \sqrt{20 - m + 24\sqrt{4-m}}\right)/8 & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

Let $\beta \in O_K$ be such that $|\beta| \geq B_m$ and $\mathrm{Re}(\beta) \geq 1$. *For an irreducible element π of O_K, if*

$$\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

is its base-β representation with $n \geq 1$ satisfying the conditions (i) and (ii) of Lemma 3.1, then $f(x)$ is irreducible in $O_K[x]$.

Proof. Suppose to the contrary that $f(x) = g(x)h(x)$, where $g(x), h(x) \in O_K[x]\backslash U(O_K)$. We first show that either $\deg g(x) \geq 1$ and $|g(\beta)| = 1$ or $\deg h(x) \geq 1$ and $|h(\beta)| = 1$. Clearly, $\deg f(x) \geq 1$ implies that $g(x)$ or $h(x)$ is a positive degree polynomial. If either $\deg g(x) = 0$ or $\deg h(x) = 0$, we may assume that $h(x) = \alpha \in O_K$. Then $f(x) = \alpha g(x)$ and so $\pi = \alpha g(\beta)$. Since π is irreducible and $\alpha \notin U(O_K)$, we obtain $g(\beta) \in U(O_K)$ and thus, $|g(\beta)| = 1$. Otherwise, both $\deg g(x) \geq 1$ and $\deg h(x) \geq 1$. Since $\pi = g(\beta)h(\beta)$ and using the irreducibility of π again, we deduce that either $g(\beta)$ or $h(\beta)$ is a unit. Hence, either $|g(\beta)| = 1$ or $|h(\beta)| = 1$.

We may assume without loss of generality that $\deg g(x) \geq 1$ and $|g(\beta)| = 1$. We next show that

$$|\beta| - \frac{1 + \sqrt{1 + 4A_m|\beta|}}{2} \geq 1, \qquad (3.1)$$

where A_m is defined as in Eq. (2.1). We treat two possible cases.

Case 1: $m \not\equiv 1 \pmod 4$. Since $|\beta| \geq B_m$, we get

$$|\beta|^2 - \frac{(6+\sqrt{1-m})|\beta|}{2} + 2 = \left(|\beta| - \frac{6+\sqrt{1-m}+\sqrt{5-m+12\sqrt{1-m}}}{4}\right) \cdot \left(|\beta| - \frac{6+\sqrt{1-m}-\sqrt{5-m+12\sqrt{1-m}}}{4}\right) \geq 0, \text{ implying}$$

$$(2|\beta|-3)^2 \geq 1 + 2\sqrt{1-m}|\beta|.$$

This shows that

$$|\beta| - \frac{1+\sqrt{1+4A_m|\beta|}}{2} = |\beta| - \frac{1+\sqrt{1+2\sqrt{1-m}|\beta|}}{2} \geq 1.$$

Case 2: $m \equiv 1 \pmod 4$. Since $|\beta| \geq B_m$, we obtain

$$|\beta|^2 - \frac{(12+\sqrt{4-m})|\beta|}{4} + 2 = \left(|\beta| - \frac{12+\sqrt{4-m}+\sqrt{20-m+24\sqrt{4-m}}}{8}\right) \cdot \left(|\beta| - \frac{12+\sqrt{4-m}-\sqrt{20-m+24\sqrt{4-m}}}{8}\right) \geq 0,$$

showing

$$(2|\beta|-3)^2 \geq 1 + \sqrt{4-m}|\beta|.$$

It follows that

$$|\beta| - \frac{1+\sqrt{1+4A_m|\beta|}}{2} = |\beta| - \frac{1+\sqrt{1+\sqrt{4-m}|\beta|}}{2} \geq 1.$$

From both cases, we obtain Eq. (3.1).

Now we have $\deg g(x) \geq 1$ and it can be expressed in the form

$$g(x) = \varepsilon \prod_i (x-\gamma_i),$$

where $\varepsilon \in O_K$ is the leading coefficient of $g(x)$ and the product is over the set of complex zeros of $g(x)$. By Theorem 2.2, we have $|\alpha_i| \leq A_m|\beta|$ for all $i \in \{0,1,\ldots,n\}$. It follows from Lemmas 3.1-3.2 that any complex zero γ of $g(x)$ satisfies either $\mathrm{Re}(\gamma) < 0$ or

$$|\gamma| < \frac{1+\sqrt{1+4A_m|\beta|}}{2}. \quad (3.2)$$

(In the case $\deg f(x) = 1$, we have either $\gamma = 0$ or $\mathrm{Re}(\gamma) < 0$). If $\mathrm{Re}(\gamma) < 0$, then $|\beta-\gamma| \geq \mathrm{Re}(\beta-\gamma) = \mathrm{Re}(\beta) - \mathrm{Re}(\gamma) > 1$. In the latter case, we obtain by Eqs. (3.1)-(3.2) that

$$|\beta-\gamma| \geq |\beta|-|\gamma| > |\beta| - \frac{1+\sqrt{1+4A_m|\beta|}}{2} \geq 1.$$

Since $|\varepsilon| \geq 1$, we get

$$1 = |g(\beta)| = |\varepsilon| \prod_i |\beta-\gamma_i| \geq \prod_i |\beta-\gamma_i| > 1,$$

which is a contradiction. This completes the proof. □

Note that O_K is a unique factorization domain for $m \in \{-1,-2,-3,-7,-11\}$. Thus, an irreducible element is a prime element in O_K. This shows that Theorem 1.2 can be extended to any imaginary quadratic field by Theorem 3.3.

By applying Theorem 3.3, we can find irreducible polynomials in $O_K[x]$ as the following examples.

Example 3.4. Let $K = \mathbb{Q}(\sqrt{-2})$ and $\beta = 4$. Then $|\beta| = 4 > \left(6+\sqrt{3}+\sqrt{7+12\sqrt{3}}\right)/4 = B_{-2}$ and $\mathrm{Re}(\beta) = 4 > 1$. Let $\pi = 2615 +$

$1281\sqrt{-2} \in O_K$. Since $N(\pi) = 10120147$ is a rational prime, we deduce that π is an irreducible element. By Eq. (2.5), we have

$$\pi = (2+\sqrt{-2})\beta^5 + (3+\sqrt{-2})\beta^4 - 3\beta^3 - 2\beta + (-1+\sqrt{-2})$$

is its base-β representation with $n = 5 > 1$ satisfying the conditions (i) and (ii) of Lemma 3.1. By Theorem 3.3, we obtain that

$$f(x) = (2+\sqrt{-2})x^5 + (3+\sqrt{-2})x^4 - 3x^3 - 2x + (-1+\sqrt{-2})$$

is irreducible in $O_K[x]$. Note that it is irreducible over K because O_K is a unique factorization domain.

Example 3.5. Let $K = \mathbb{Q}(\sqrt{-15})$ and $\beta = 5$. Then $|\beta| = 5 > \left(12 + \sqrt{19} + \sqrt{35 + 24\sqrt{19}}\right)/8 = B_{-15}$ and $\mathrm{Re}(\beta) = 5 > 1$. Let $\pi = 56759 + 16248\sigma_{-15} \in O_K$. Since $N(\pi) = 5199794329$ is a rational prime, we deduce that π is an irreducible element. One can verify that

$$\pi = (3+\sigma_{-15})\beta^6 + 4\beta^5 + (-5+\sigma_{-15})\beta^4 + 4\beta^3 + \beta + (4-2\sigma_{-15}) \quad (3.3)$$

is its base-β representation with $n = 6 > 1$ satisfying the conditions (i) and (ii) of Lemma 3.1. By Theorem 3.3, we deduce that

$$f(x) = (3+\sigma_{-15})x^6 + 4x^5 + (-5+\sigma_{-15})x^4 + 4x^3 + x + (4-2\sigma_{-15})$$

is irreducible in $O_K[x]$.

From Example 3.4, we have that

$$C' = \left\{x + y\sqrt{-2} \mid x, y = 0, 1, 2, 3\right\}.$$

It is important to emphasize that we cannot apply Theorem 3 in [11] to conclude the irreducibility of the polynomial $f(x)$. This is because the representation of π in Eq. (2.5) is not a base-$\beta(C')$ representation, even though $|\beta| = 4 > 2 + \sqrt{3}$ and $a = 4 > 1 + \sqrt{3}$.

Similarly, in Example 3.5, we have

$$C' = \{x + y\sigma_{-15} \mid x, y = 0, 1, 2, 3, 4\}.$$

Since the representation of π in Eq. (3.3) is not a base-$\beta(C')$ representation, Theorem 4 in [11] cannot be applied to conclude the irreducibility of the polynomial $f(x)$.

4. A Generalization of the Irreducibility Criterion

Let $K = \mathbb{Q}\left(\sqrt{m}\right)$ be an imaginary quadratic field such that its ring of integers O_K is a unique factorization domain. Then $m = -1, -2, -3, -7, -11, -19, -43, -67, -163$ [13]. For a nonconstant polynomial $f(x) \in O_K[x]$, the greatest common divisor of the nonzero coefficients of $f(x)$ is called the *content* of $f(x)$, which is denoted by $c\big(f(x)\big)$. Moreover, $f(x)$ is called a *primitive polynomial* if its content is a unit. We now recall the following essential lemma from [6].

Theorem 4.1. ***(Gauss's Lemma).*** *Let R be a unique factorization domain. Let $f(x)$ and $g(x)$ be two nonzero polynomials in $R[x]$. Then*

$$c\big(f(x)g(x)\big) = uc\big(f(x)\big)c\big(g(x)\big),$$

where u is a unit in R.

It is well known that every algebraic number is of the form r/s, where r is an algebraic integer and s is a nonzero rational integer. Thus, for a nonconstant polynomial $f(x) \in O_K[x]$, if $f(x) = g_1(x)h_1(x)$, where $g_1(x)$ and $h_1(x)$ are positive degree polynomials in $K[x]$, then we may take $g_2(x) = ag_1(x)$ and $h_2(x) = bh_1(x)$ for some nonzero rational integers a and b and nonconstant polynomials $g_2(x), h_2(x)$ in $O_K[x]$. Hence $abf(x) = g_2(x)h_2(x)$ and so

$$c\big(abf(x)\big) = uc\big(g_2(x)\big)c\big(h_2(x)\big), \quad (4.1)$$

by Theorem 4.1, where $u \in U(O_K)$. Let π be a prime divisor of ab in O_K. One can see that π divides $c\big(abf(x)\big)$. Since π is a prime element of O_K, it follows from Eq. (4.1) that π divides $c\big(g_2(x)\big)$ or $c\big(h_2(x)\big)$. Thus π divides $g_2(x)$ or $h_2(x)$, yielding

$$\frac{ab}{\pi} f(x) = g_3(x)h_3(x),$$

for some nonconstant polynomials $g_3(x)$ and $h_3(x)$ in $O_K[x]$. Continuing in the same manner, we finally get $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are positive degree polynomials in $O_K[x]$. Hence, we deduce that for a nonconstant polynomial $f(x) \in O_K[x]$, if $f(x)$ is reducible over K, then it is reducible in $O_K[x]$. Indeed, if $f(x)$ is primitive, then $f(x)$ is irreducible in $O_K[x]$ if and only if $f(x)$ is irreducible over K [14].

For an imaginary quadratic field K and a nonconstant polynomial $f(x) \in O_K[x]$, we say that $f(x) = g(x)h(x)$ in $O_K[x]$ is a *proper factorization* if both $g(x)$ and $h(x)$ have a smaller degree than $f(x)$. We now proceed to the last main result, which extends Theorem 1.5 to any imaginary quadratic field K.

Theorem 4.2. *Let $K = \mathbb{Q}\left(\sqrt{m}\right)$ be an imaginary quadratic field. Let $\beta \in O_K$ and $\omega \in O_K \backslash \{0\}$ be such that* $\mathrm{Re}(\beta) \geq |\omega|$ *and $|\beta| \geq B_m(\omega)$, where $B_m(\omega) :=$*

$$\begin{cases} \Big(2(2|\omega|+1) + \sqrt{1-m} + \\ \sqrt{5 - m + 4(2|\omega|+1)\sqrt{1-m}}\Big)/4 \\ \qquad\qquad \text{if } m \not\equiv 1 \pmod 4, \\ \Big(4(2|\omega|+1) + \sqrt{4-m} + \\ \sqrt{20 - m + 8(2|\omega|+1)\sqrt{4-m}}\Big)/8 \\ \qquad\qquad \text{if } m \equiv 1 \pmod 4. \end{cases}$$

For a prime element π of O_K, if

$$\omega\pi = \alpha_n\beta^n + \alpha_{n-1}\beta^{n-1} + \cdots + \alpha_1\beta + \alpha_0 =: f(\beta)$$

is its base-β representation with $n \geq 2$ satisfying the conditions (i) and (ii) of Lemma 3.1, then $f(x)$ has no proper factorization in $O_K[x]$. Moreover,

(i) if $\delta \nmid f(x)$ for all $\delta \in O_K \backslash U(O_K)$, then $f(x)$ is irreducible in $O_K[x]$.

(ii) If O_K is a unique factorization domain, then $f(x)$ is irreducible over K.

Proof. Suppose to the contrary that $f(x)$ has proper factorization in $O_K[x]$. Then $f(x) = g(x)h(x)$ for some nonconstant polynomials $g(x)$ and $h(x)$ in $O_K[x]$, so $\omega\pi = g(\beta)h(\beta)$. Since π is a prime element, either $\pi \mid g(\beta)$ and $h(\beta) \mid \omega$ or $\pi \mid h(\beta)$ and $g(\beta) \mid \omega$. This implies that either $|\omega| \geq |h(\beta)|$ or $|\omega| \geq |g(\beta)|$. Without loss of generality, we may assume that $|\omega| \geq |g(\beta)|$.

We next show that

$$|\beta| - \frac{1 + \sqrt{1 + 4A_m|\beta|}}{2} \geq |\omega|, \quad (4.2)$$

where A_m is defined as in Eq. (2.1). Consider two possible cases.

Case 1: $m \not\equiv 1 \pmod 4$. Since $|\beta| \geq B_m(\omega)$, we have

$$|\beta|^2 - \frac{\left[2(2|\omega|+1)+\sqrt{1-m}\right]|\beta|}{2} + (|\omega|^2+|\omega|) = \left(|\beta| - \left(2(2|\omega|+1)+\sqrt{1-m} + \sqrt{5-m+4(2|\omega|+1)\sqrt{1-m}}\right)/4\right) \cdot \left(|\beta| - \left(2(2|\omega|+1)+\sqrt{1-m} - \sqrt{5-m+4(2|\omega|+1)\sqrt{1-m}}\right)/4\right) \geq 0,$$

implying $[2|\beta| - (2|\omega|+1)]^2 \geq 1 + 2\sqrt{1-m}|\beta|$. This shows that

$$|\beta| - \frac{1+\sqrt{1+4A_m|\beta|}}{2} = |\beta| - \frac{1+\sqrt{1+2\sqrt{1-m}|\beta|}}{2} \geq |\omega|.$$

Case 2: $m \equiv 1 \pmod 4$. Since $|\beta| \geq B_m(\omega)$, we have

$$|\beta|^2 - \frac{\left[4(2|\omega|+1)+\sqrt{4-m}\right]|\beta|}{4} + (|\omega|^2+|\omega|) = \left(|\beta| - \left(4(2|\omega|+1) + \sqrt{4-m} + \sqrt{20-m+8(2|\omega|+1)\sqrt{4-m}}\right)/8\right) \cdot \left(|\beta| - \left(4(2|\omega|+1)+\sqrt{4-m} - \sqrt{20-m+8(2|\omega|+1)\sqrt{4-m}}\right)/8\right) \geq 0,$$

showing $[2|\beta| - (2|\omega|+1)]^2 \geq 1 + \sqrt{4-m}|\beta|$. It follows that

$$|\beta| - \frac{1+\sqrt{1+4A_m|\beta|}}{2} = |\beta| - \frac{1+\sqrt{1+\sqrt{4-m}|\beta|}}{2} \geq |\omega|.$$

From both cases, we obtain Eq. (4.2).

Now, we have that $\deg g(x) \geq 1$, so it can be expressed in the form

$$g(x) = \varepsilon \prod_i (x-\gamma_i),$$

where $\varepsilon \in O_K$ is the leading coefficient of $g(x)$ and the product is over the set of complex zeros of $g(x)$. By Theorem 2.2, we have $|\alpha_i| \leq A_m|\beta|$ for all $i \in \{0, 1, \ldots, n\}$. It follows from Lemma 3.1 that any complex zero γ of $g(x)$ satisfies either $\mathrm{Re}(\gamma) < 0$ or

$$|\gamma| < \frac{1+\sqrt{1+4A_m|\beta|}}{2}. \tag{4.3}$$

If $\mathrm{Re}(\gamma) < 0$, then $|\beta-\gamma| \geq \mathrm{Re}(\beta-\gamma) = \mathrm{Re}(\beta) - \mathrm{Re}(\gamma) > |\omega|$. In the latter case, we obtain by Eqs. (4.2)- (4.3) that

$$\begin{aligned} |\beta-\gamma| &\geq |\beta| - |\gamma| \\ &> |\beta| - \frac{1+\sqrt{1+4A_m|\beta|}}{2} \\ &\geq |\omega|. \end{aligned}$$

From both cases, we deduce that

$$\begin{aligned} |\omega| &\geq |g(\beta)| \\ &= |\varepsilon| \prod_i |\beta-\gamma_i| \\ &\geq \prod_i |\beta-\gamma_i| \\ &> |\omega|, \end{aligned}$$

which is a contradiction.

Finally, it is clear that (i) holds. By the explanation mentioned earlier, we can infer that if $f(x)$ has no proper factorization in $O_K[x]$, where O_K is a unique factorization domain, then it is irreducible over K. This proves (ii). □

We end this paper by the following examples, which illustrate the use of Theorem 4.2.

Example 4.3. Let $K = \mathbb{Q}(\sqrt{-1})$, $\beta = 12 - 9i$, $\omega = 1 + i$, and $\pi = -82212 - 21517i$. Then $\mathrm{Re}(\beta) = 12 > \sqrt{2} = |\omega|$ and $|\beta| = 15 > \left(2(2\sqrt{2} + 1) + \sqrt{2} + \sqrt{6 + 4(2\sqrt{2} + 1)\sqrt{2}}\right)/4 = B_{-1}(1 + i)$. Note that O_K is a unique factorization domain and π is a prime element because $N(\pi) = 7221794233$ is a rational prime. One can verify that

$$\begin{aligned}\omega\pi &= -60695 - 103729i \\ &= (2 + i)\beta^4 + 2\beta^3 + (1 - i)\beta^2 - \\ &\quad i\beta + (1 - i)\end{aligned}$$

is its base-β representation with $n = 4 > 2$ satisfying the conditions (i) and (ii) of Lemma 3.1. By Theorem 4.2, we obtain that

$$f(x) = (2+i)x^4 + 2x^3 + (1-i)x^2 - ix + (1-i)$$

has no proper factorization in $O_K[x]$ and so is irreducible over K by Theorem 4.2(ii). Since $-i$ is a unit in $\mathbb{Z}[i]$, it follows that $f(x)$ is primitive and so is irreducible in $O_K[x]$ by Theorem 4.2(i).

Example 4.4. Let $K = \mathbb{Q}(\sqrt{-19})$, $\beta = 17 + 5\sigma_{-19}$, $\omega = 2$, and $\pi = -59062661 + 16019945\sigma_{-19}$. Then $\mathrm{Re}(\beta) = 19.5 > 2 = |\omega|$ and $|\beta| = \sqrt{499} > (20 + \sqrt{23} + \sqrt{39 + 40\sqrt{23}})/8 = B_{-19}(2)$. Note that O_K is a unique factorization domain and π is a prime element because $N(\pi) = 3825410532642401$ is a rational prime. One can verify that

$$\begin{aligned}\omega\pi &= -118125322 + 32039890\sigma_{-19} \\ &= 22\beta^5 + 6\beta^4 + 4\beta^3 + 10\beta^2 + 8\beta + 20\end{aligned}$$

is its base-β representation with $n = 5 > 2$ satisfying the conditions (i) and (ii) of Lemma 3.1. By Theorem 4.2, we conclude that

$$f(x) = 22x^5 + 6x^4 + 4x^3 + 10x^2 + 8x + 20$$

has no proper factorization in $O_K[x]$ and so is irreducible over K by Theorem 4.2(ii). Note that $f(x)$ is reducible in $O_K[x]$ because $f(x) = 2(11x^5 + 3x^4 + 2x^3 + 5x^2 + 4x + 10)$.

By substituting $\omega \in U(O_K)$ into Theorem 4.2, we obtain that the polynomial $f(x)$ has no proper factorization in $O_K[x]$. Since π is also irreducible, the polynomial $f(x)$ is primitive by Theorem 3.3, and thus, it is irreducible in $O_K[x]$. This means that Theorem 4.2 is a generalization of Theorem 3.3 by considering $\omega\pi$ instead of π, where $\omega \in O_K\backslash\{0\}$ and π is a prime element.

5. Conclusion

Let K be an imaginary quadratic field with the ring of integers O_K. In this work, we construct the base-β representation for any nonzero element of O_K. Moreover, we establish further irreducibility criteria for polynomials in $O_K[x]$. These results extend the existing results in the literature.

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